Forecasted Learning*

Raquel Lorenzo Vidal† Paul O. Richter‡

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Abstract

We consider a Bayesian decision maker (DM) who, before making a decision, needs to allocate her limited attention across news sources with different biases. When choosing what news to read, the DM expects to receive some additional information in the future beyond her control. We show that the expectation of future information may affect the DM's optimal learning decision in several ways. In particular, it can rationalize both the choice of news that reinforce or weaken one's prior (own and opposite-biased learning). The DM chooses own-biased learning when she is very certain of her action and, as long as the additional information is sufficiently powerful, opposite-biased learning when she is moderately certain. On the other hand, a very uncertain DM might want to make her choice of news dependent on the bias of the additional information. Applying our rational framework to study how expected future social interactions can impact peoples news consumption decisions, we show that people may (mis)coordinate the type of news they read with respect to their social group.

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†UPF & BSE, E-mail: raquel.lorenzo@upf.edu
‡UPF & BSE, E-mail: paul.richter@upf.edu

1 Introduction

Individuals are constantly learning, gathering information to make better decisions. From choosing what news source to read, to choosing what type of market research to perform, people make choices on the structure of their informational sources everyday. A growing stream of literature studies how to optimally choose between different sources of information to better understand these situations that arise due to, among other reasons, limited attention, costly information, etc. However, most papers in the literature on information choice have been missing a very natural feature of many learning environments: agents often only have partial control over the information they receive[1].

In many situations agents have some control over their information. For instance, individuals can choose where to get their news, knowing before-hand the ideological bias of different news-sources. But often they cannot fully decide what type of information to receive. For instance, peoples' social circle tend to share information with them.

In this paper, we study agents' learning decisions and, more specifically, choices of news bias, when they forecast receiving information, which is beyond their control, sometime in the future. We ask the question of how expecting to receive information beyond one's control may change the optimal choice of news bias. In addition, we study how the features of the additional information and of the menu of media outlets affect the decision, as well as how this depends on the agent's prior belief. In our application, we also explore how the interaction with others, in particular, hearing about news that others read, changes individual and collective choices of media outlets.

Although we focus mainly on media consumption, our results are relevant to many other learning environments. For example, before deciding on treatments, doctors can choose what tests to run, knowing the rates of false positive and false negatives of each test; CEO's can test the profitability of a new technology in different ways, from focus groups to field

[1] We differentiate our contribution from other recent work such as the working paper of Brooks et al. [2023] in the literature review below.

experiments. In these information choice contexts, the agents also expect to receive some future information which is beyond their control. The doctor will expect different illnesses to have different observable symptom progressions and the CEO may expect to observe consumer responses to competitors who test a similar technology. This second feature of the problem is what many other models of optimal learning have been missing: the expectation of future information not controlled by the agents.

In order to study this issue, we propose a simple model with two states of the world and a Bayesian decision-maker (DM) who wants to choose an action that matches the state. Before choosing the action, she can choose to seek information from one of two news sources (signals), each biased towards a specific action. On top of this, in between her choice of source and her choice of action, she receives an additional message (signal realization) from an exogenous source, of known structure. To address the questions above, we analyze the DM's optimal choice of news source depending on the structure of the available news sources, the additional information and the DM's prior belief.

We find that this natural change to a standard model of allocation of attention to biased sources generates rich changes to the standard predictions.[2] First, we are able to rationalize learning from sources that weaken one's prior (opposite-biased learning). The mechanism that explains this is different from previous literature, which used dynamic problems of allocation of attention. Second, the expectation of future information leads to optimal learning strategies that depend in interesting ways on the DM's prior. For instance, a very certain DM may choose own-biased learning, while a moderately certain DM finds opposite-biased learning optimal and an uncertain one reads the news which bias coincides with the bias of the additional information.

We show that expectations on future information play an essential role in determining the optimal information choice. This is the case even when agents have not yet received

[2]Many papers in the current literature studying allocation of attention assume that individuals have full control over the signals that they process. On the other hand, many papers looking at learning in groups or learning from other outside sources do not consider allowing the agent to have some control/choice of the signal structure. For specific papers see the literature review below.

the additional information at the time of the decision. The mere expectation of receiving it, may be enough to affect it. We highlight the important role of expectations of future information in explaining agents' optimal learning decisions, by showing that, as long as one source of information is not objectively better (better for any prior) than the other, there always exists an exogenous signal that can change the DM's optimal choice between two sources of information, no matter what her prior is.

We also find sufficient conditions for indifference as well as for either choosing sources that reinforce (own-biased learning) or weaken (opposite-biased learning) one's prior to be optimal. If a DM is extremely sure about the state, she is indifferent between any source of information, since none of them will be sufficiently powerful to change her action. If she is very sure but can learn something valuable, she will find it optimal to choose own-biased learning. Under some conditions on the exogenous source, a moderately biased DM's optimal choice will be opposite-biased learning. The choice of a central DM is more involved. If the exogenous source is not very informative, the DM will choose "as if" she had full control over her information, that is, own-biased learning. If the exogenous source is very informative, the DM will be indifferent about what news to choose. Otherwise, if the exogenous source is as informative as the information available to the DM, she will choose a source with the same type of bias as the exogenous information. Finally, we provide a full characterization of the DM's optimal choice of source, depending on her prior, for a broad class of exogenous information.

In many contexts, the information structure of the additional information one expects to receive is the outcome of decisions made by other agents. In that case, the structure of the future signal may be endogenous to the choice made by the agent. This observation leads to several meaningful applications of our framework that endogenize the additional information. We go over a specific application to the choice of media bias with a strategic interaction, in order to better understand how people learn within societies. Instead of one, we consider two Bayesian players with (possibly different) priors about the state. Players have access to two types of news sources, left-biased and right-biased. They simultaneously choose what type of source they want to look at and, afterwards, they (mechanically) share the information that they obtained with each other. After updating their beliefs, using both their own and their peer's information, each player chooses an action (left or right) and obtain their payoffs, which are maximized when the action matches the state. Their objective is to learn the state of the world but, unlike in standard models, they do not have full control over the information that they acquire. Instead, they are aware that they will receive a some additional information chosen by the other player.

In accordance with the results of our baseline model, we show that the choices of news bias will sometimes depend on the expectation of others' news choice. Therefore, opposite-biased learning (choosing a media source biased against one's prior) can be optimal for individual learning. We find that agents may have incentives to coordinate in their choice of news and we derive a set of priors for which there exist equilibria with one or both agents learning from an opposite biased source. Using this result, we show basic comparative statics for how this set changes with the accuracy of the available media outlets.

The remainder of the paper is organized as follows. In Section 2 we connect our results to the relevant literature. Section 3 illustrates the main insights of the paper with a simple example. In Section 4 we introduce our model and the key definitions. In Section 5 we present our main results and in Section 6 we propose an application to learning with social interaction. We conclude in Section 7.

2 Literature Review

Theorists have been studying the comparison of different information structures for a very long time. Famously, Blackwell (1958) gives conditions under which one information source is better than another, regardless of preferences. However, it was not until recently, that this literature started to account for the effect of the external information on the choice.

Börgers et al. [2013a] discuss complementarity between signals, where certain signals may increase each others value, regardless of preferences. Brooks et al. [2023] develop ways to compare signals, similar to Blackwell, however, robust to the existence of additional, possibly complementary signals. Our analysis differs in two important ways. Firstly, we focus on signal structures that are independent conditional on the state. We show that there are very rich decision patterns, without including additional complementarities coming from signal correlation. Secondly, the optimal choice in our analysis depends on the prior beliefs of the agent. In contrast, Brooks et al. [2023], Börgers et al. [2013a] and Blackwell focus on contexts where one signal dominates the other regardless of prior belief. Our analysis focuses on the choice of media bias between two sources of similar quality, while their analysis focuses on showing when one signal is clearly higher quality than another for all preferences.

In this paper, we focus on choices of a Bayesian agent between biased information sources, such as news sources. Therefore, our results contribute to the literature of Bayesian learning with information choice. Papers showing the optimality of consuming an own-biased medium for a Bayesian agent in a static setting are Calvert [1985]; Suen [2004]; Burke [2008]; Gentzkow and Shapiro [2006]; Meyer [1991]; Mullainathan and Shleifer [2005] and Zhong [2022]. As in our paper, Calvert [1985]; Suen [2004]; Burke [2008] Gentzkow and Shapiro [2006] and Zhong [2022] consider a fully rational agent whose objective is to learn the state, but they do not account for any additional information that the agent may expect to receive. Our results are fully consistent with their finding if no additional information is expected. However, when including this new feature, we find that opposite-biased learning can also be optimal. Mullainathan and Shleifer [2005] rationalize own-biased learning by assuming that agents obtain a disutility from getting information against their prior. In our work, the structure of the information, does not enter the agent's utility directly: only indirectly through learning. On the contrary, Oliveros and Várdy [2015] find that when voters with the option of abstention make informational choices, central sources might be more attractive. Unlike theirs, our model only considers two possible actions.

Recent work using dynamic models, shows that it can be optimal for a Bayesian agent with costly attention to multi-home or learn from opposite-biased sources (Che and Mierendorff [2019]; Nikandrova and Pancs [2018]; Mayskaya [2020]; Liang et al. [2022] and Georgiadis-Harris [2023]). In Che and Mierendorff [2019]; Nikandrova and Pancs [2018]; Mayskaya [2020] and Liang et al. [2022], agents face an optimal stopping problem: they decide what sources to sample signals from until they choose to stop and make a decision. In Georgiadis-Harris [2023], instead, the decision maker lacks control over the timing of her action. Unlike in our work, all of them rely on a dynamic mechanism to justify multi-homing and opposite-biased learning. Moreover, they consider agents with no strategic concerns, in the sense that they choose without the need to reason about others. In our main application, we include social interaction and the corresponding strategic considerations.

Another related literature is on rational inattention (see Maćkowiak et al. [2023] for a recent review). There, the cost of attention is proportional to the change from the prior belief to the posterior, after updating on the signal realization. In contrast, in our model, agents have a fixed and limited amount of attention that they can allocate to available sources.

Papers that consider exogenous manipulations of beliefs, which affect agents informational choices are Gossner et al. [2021]; Liang et al. [2022]; Dworczak and Pavan [2022]; Laclau et al. [2017] and Kolotilin et al. [2017]. Gossner et al. [2021] and Liang et al. [2022] study the problem of optimal dynamic allocation of attention among sources providing information about different items, when attention can be exogenously manipulated. The two main differences with our work are that: i) they consider an optimal stopping problem, while our game has a given stopping time; ii) they study how manipulating attention at a given point in time affects consequent learning choices, while we study how expected manipulations of attention in the future affect current learning choices. Laclau et al. [2017]; Kolotilin et al. [2017] and Dworczak and Pavan [2022] study the problem of a persuader who is uncertain about the beliefs of the receiver or the additional information that the receiver might obtain. These contributions are *conceptually* related to ours in that a decision maker takes into account a distribution of posteriors when making informational choices (in their case, what signal to share with the receiver). However, in our work the objective of the decision-maker is different: instead of persuading to choose a specific (state-independent) action, the objective is to choose a state-dependent action.

Our work also relates to the literature studying signal's distortion towards a state and how it affects decision processes. In media contexts, like ours, it can be interpreted as news-bias (Che and Mierendorff [2019]). But, for example, Masatlioglu et al. [2023] refer to this distortion as skewness, and study individuals' preferences on skewness towards negative vs positive outcomes. In the context of firm innovation, Gans [2023] shows that incumbents have different optimal signal distortion compared to entrants when testing new, possibly disruptive, technologies.

In our application, we also contribute to the literature on (Bayesian) learning from others. One stream of this literature is that on herding (Banerjee [1992]; Bikhchandani et al. [1998]; Smith and Sørensen [2000]), which highlights the role of peers' actions as a source of public information from which agents learn. More recent literature studying opinion dynamics and learning in social networks (for a review, see Acemoglu and Ozdaglar [2011]) is also closely related to our application. Different from this paper, this literature often assumes that agents have either full or no control over the information they receive.

3 Illustrative Example

We first go over a simple illustrative example to introduce our setting as well as to show how it can produce different results than models in the current literature.

Consider Ann, who needs to choose whether to vote left (L) or right (R) tomorrow. The state of the world could either be left or right (the state $\theta \in \{L, R\}$). If her vote matches the state of the world she receives a normalized value of 1, and if her vote does not match the state of the world she receives a value of 0.

Before informing herself further, Ann has a prior belief that she is in the left state of the world with probability 0.6. Since she believes that the left state of the world is more likely, we say that she is *left-biased*.

Today, Ann can choose to read one of two news sources which can both tell her either "left" or "right".[3] One of them is left-biased and the other is right-biased. In the left state, the left biased news source always tells her "left". However, in the right state the left biased source tells her "right" only half the time, and left the other half. Similarly, the message from the right biased source always matches the state in the right state, however, the right-biased source reports "right" with a 50% probability in the left state.

This means that the message of right biased news will be match the state with higher probability in the right state, and the message of the left biased news will match the state with higher probability in the left state. However, seeing the left-biased source saying "right" is very strong evidence of the best action being R. In fact, she can be sure that R is the best action. The analogous holds for right-biased news.[4]

Ann is completely aware of how the two news sources generate their messages. She chooses a news source that maximizes the value of her vote tomorrow, given her prior on the underlying state of the world and the way the she expects sources to report in each state. More specifically, given the structure of the news sources in each state she can calculate her probability of matching the state, given each state. Then she weights these probabilities by her prior belief of being in each state.

If she made this choice in isolation, given her prior and the messaging structure of the sources, the left-biased source would be her optimal choice. We call this *own-biased learning*, since she chooses the source whose bias is in the same direction as her prior. This prediction is consistent with the findings of previous literature on learning from biased sources.

However, Ann does not learn in isolation. She has a friend, Bob, who often shares the

[3]This can be thought of observing the realization of one of two signals.

[4]This binary message structure with the possibility of the state being know with certainty is used in many papers, including Che and Mierendorff [2019] and Gans [2023].

news that he read with her. The two friends are meeting tomorrow before the election and Ann knows that Bob tends to read right-biased news. Assuming that, given the state, the message that Bob obtains from the news he reads is independent of the message that Ann would get by reading the same type of news (otherwise Ann's optimal choice is trivial), what would be Ann's optimal choice of news bias?

One way of interpreting Ann's revised problem is to view her as choosing between the bundle of a left and a right source and the bundle of two right sources. However, the right source from Bob is not under her control. There exists another way to look at her problem, that focuses on only the sources she can choose between. One can solve her choice problem by considering each possible posterior after updating from only Bob's signal, we call these interim-posterior beliefs. Her optimal signal is the signal which gives her the highest chance of matching the state, across the interim-posterior beliefs of Bob's news source.[5]

If Bob's right-biased source sends the message "left", then Ann would be certain that she is in the left state. At this posterior she knows the state, and the source she picks would not further improve her vote's accuracy. However, if Bob's source sends the message "right", Ann's updated belief would be that the right state is more likely. At this interim-posterior, she would prefer the right biased source. Therefore, expecting to receive information from a right-biased source tomorrow, Ann's optimal choice today is to seek information from a right-biased source. Since Ann is left-biased, this means that now it is optimal for her to choose an *opposite-biased learning* strategy.[6]

This simple example showcases how expecting to obtain additional information in the future may change someone's optimal learning decisions and break the standard prediction from the literature on learning from biased sources that a Bayesian agent will find it optimal to seek information from the source that is biased in the direction of her prior (own-biased learning).

[5]This perspective is formalized in Observation 1 below

[6]For simplicity, we use signals that may reveal the state with certainty in this example. In our analysis we relax the signal structure to show that opposite biased learning also happens with signals that do not have this perfect revelation feature.

4 Learning Model

A Bayesian decision-maker (DM) must choose from two actions, A^L or A^R, trying to match an unknown state $\theta \in \{L, R\}$. The payoff of choosing action $A^x \in \{A^L, A^R\}$ is 1 if the action matches the state, $x = \theta$ and is 0 if the action does not match the state, $x \neq \theta$. The DM has a prior belief about the probability of the state being R, which is denoted by $p_0 \in [0, 1]$. The DM has access to two news sources (signal structures) that she can choose from: a *right-biased* source σ^R and a *left-biased* source σ^L. Moreover, she receives information from an exogenous source σ_e. This represents any additional information that the DM expects to receive which is beyond her control. It could be news shared by a friend, conversations heard in the office, information obtained by experience, advertising, etc.

Information from a combination of different sources can also be interpreted as one aggregate source. The important distinction is between sources which the DM can choose (σ^L and σ^R) and the exogenous source over which the DM does not have any control (σ_e). Focusing on the choice between a left and right biased signal shows how exogenous signals can shift the direction of the bias of the signal chosen. [7]

Sources of information. A news source σ^x, can send two possible messages, l or r. Each source is characterized by two parameters: the probability of sending message l when the state is L, $\pi^x(l|L)$, and the probability of sending message r when the state is R, $\pi^x(r|R)$ as shown in Table 1 below. Formally, a news source is a binary signal structure, that is, a distribution of binary messages conditional on the state.[8] With binary signals, one message will always cause the posterior to weight state L more than the prior. We label l as the signal realization that causes the DM to update her prior to believe state L is more likely. Therefore w.l.o.g., we can assume that $\pi^x(l|L) \geq 1 - \pi^x(r|R)$.

[7] Although, we focus on this choice for expositional reasons, this framework also can be easily extended and applied to choices between a larger set of signals, with varying biases. Certain propositions below are shown to hold in a more general setting.

[8] Many of our key results generalize to non-binary signal structures. We will point those out along the paper.

State/Message	l	r
$\theta = L$	$\pi^x(l\|L)$	$1 - \pi^x(l\|L)$
$\theta = R$	$1 - \pi^x(r\|R)$	$\pi^x(r\|R)$

Table 1: Signal structure of a binary signal σ^x

In addition, whenever $\pi^x(l|L) = 1-\pi^x(r|R)$, σ^x the source is completely uninformative, in the sense that after receiving any message from such source the DM's posterior will be equal to her prior. On the other hand, the source is fully revealing in both states when $\pi^x(l|L) = 1$ and $\pi^x(r|R) = 1$. In order to avoid uninteresting limit cases, we will not consider perfectly informative nor perfectly uninformative sources in our analysis.

Unless expressed otherwise, we will be focusing on the case where the signals are independent of each other, conditional on the state. In reality, the messages of different sources might be correlated. However, here we are abstracting from this to isolate the effect of the sources' bias and informativeness.

In order to investigate choices of news-bias, we formalize a definition of bias for binary signals. This definition is compatible with discussions of binary signals and news bias found in other papers in the literature such as Che and Mierendorff [2019], and has clear interpretations as discussed below.

Definition 1 *A source σ^x is:*

i) right-biased if $\pi^x(l|L) < \pi^x(r|R)$,

ii) left-biased if $\pi^x(l|L) > \pi^x(r|R)$,

iii) unbiased if $\pi^x(l|L) = \pi^x(r|R)$.

Under this definition a message from a left-biased source is more likely to match the true state of the world when the state is left. A message from a right biased source is more likely to match the true state in the right state. This fits with the intuition that a source is more likely to be correct when the state of the world matched their bias.

Another way to interpret this definition is from the perspective of someone who believes both states of the world are equally likely. Someone with this central prior, would expect the right biased source be more likely to send a right message than a left message. Similarly, they would expect a left biased source to be more likely to send a left message than a right message.

In our analysis below, one of our aims is to determine the effect of the exogenous information on the bias of the chosen news source. To focus on bias, we often compare symmetric signals. The following definition formalizes what we mean by "symmetric" signals.[9]

Definition 2 *Two sources σ^x and σ^y are symmetric if $\pi^x(l|L) = \pi^y(r|R)$ and $\pi^x(r|R) = \pi^y(l|L)$.*

When σ^L and σ^R are symmetric, the left-biased source is as correct in the left state as the right-biased source is in the right state and as correct in the right state as the right-biased source in the left state. When the two states are equally likely, the left-biased source sends message l (r) with the same probability as the right-biased source sends message r (l). Note that if σ^L and σ^R are symmetric, for any distribution of the states, the left-biased source is more likely to send message l and the right-biased source is more likely to send message r.

Timing, information and beliefs. First, Nature chooses the state of the world θ and the DM's prior p_0. Then, the DM chooses a news source $\sigma_i \in \{\sigma^R, \sigma^L\}$. After choosing her news σ_i, the DM observes two (independent) messages: one from the source she chose, σ_i, and one from an exogenous source, σ_e, which can have the same structure as σ^L or σ^R or not. Finally, after updating her beliefs with the information received from each source, the DM chooses an action A^x and the payoff is realized.

[9]Note the difference with respect to Masatlioglu et al. [2023]'s definition of symmetry, which refers to unbiasedness in our language.

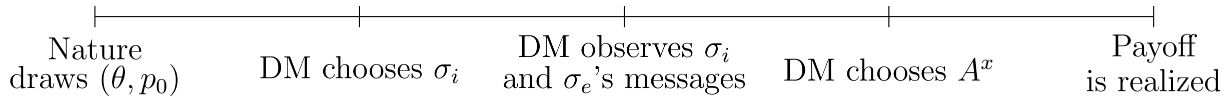

Figure 1: Timing of the learning process

When choosing σ_i, the DM is uncertain about the state (holds a prior p_0) and knows the structure of σ_e. Namely, she has not received a message yet, but knows the probabilities of receiving each message conditional on the state ($\pi^e(l|L)$ and $\pi^e(r|R)$). Moreover, she knows that she will observe the realized message after making her choice of source but before choosing the payoff-relevant action, A^x. The timing of the choice of source is key. If the DM observes the exogenous source's message before making her own source choice, then she would just choose a source based on her updated belief. In our setting however, the DM's information choice occurs before they receive the message from the exogenous source. Therefore, her optimal choice of source is dependent on how well σ_i complements the information expected from σ_e.[10]

We will denote the updated belief of the DM after observing message s from source σ^x by $p(s^x)$ and the updated belief of the DM after observing message s from source σ^x and message m from source σ^y by $p(s^x, m^y)$. Both of them represent the probability that the state is R.

5 Analysis of General Binary Setting

We are interested in the possible effects that expectations of additional information can have on the choice news bias. One approach is to view the chosen source and the exogenous source as one aggregate source of information (for example in the binary setting this aggregate source would have 4 possible messages: ll, lr, rl, rr). However, this perspective makes

[10]Through the lens of Börgers et al. [2013b]'s framework on signal complementarity and substitutability, all the signals in our environment are substitutes. However, some are more complementary than others in that the "consumption" of one signal lowers the marginal value of the next signal less for some signals than for others.

it difficult to compare how different exogenous sources impact the optimal choice of bias. Instead, we split up the problem in two parts. One part looks at all the possible posteriors that can be reached after observing messages from just the exogenous source. We refer to posteriors obtained after updating using only the exogenous source as interim posteriors. The second part looks at the difference in expected value of the left and right biased source in isolation for all possible beliefs. This second part captures how much more the DM prefers one bias over the other, without any additional signal and is analogous to exercises in the literature that omit the exogenous source. One can then view the DM's problem as taking the weighted average of the preferences for bias across the possible interim posteriors from the exogenous source. In other words, the addition of the exogenous information causes the DM to evaluate the left and right-biased sources across a distribution of possible beliefs induced by the exogenous source. When there is no additional information, the DM would just compare the sources at her prior. Observation 1 below formalizes this perspective.

Observation 1 *For a given exogenous source of information σ_e:*

$$EU(\sigma^L, \sigma_e|p_0) - EU(\sigma^R, \sigma_e|p_0) \geq 0 \iff \sum_{s \in \{l,r\}} \mathbb{P}(s^e|p_0)\Big(EU(\sigma^L|p(s^e)) - EU(\sigma^R|p(s^e))\Big) \geq 0$$

where $\mathbb{P}(s^e|p_0) = p_0\pi^e(s^e|R) + (1-p_0)\pi^e(s^e|L)$ is the probability that the DM attaches to the exogenous source σ_e sending message s^e. $EU(\cdot, \cdot|p_0)$ refers to the expected value of receiving information from a bundle of sources, for a DM with prior p_0 (just after Nature moves). $EU(\cdot|p)$ refers to the expected value of receiving information from one source for a DM with prior p (just after Nature moves and without expecting to receive any additional information). Again, this observation shows that to compare news sources, the DM can use a weighted average of the difference in expected values of the two news sources, σ^L and σ^R, across all of the interim beliefs induced by the exogenous source. In other words, the decision process can be framed as follows: first, given her prior belief, p_0, the DM forecasts the probability of receiving each message from the exogenous source, $\mathbb{P}(s^e|p_0)$, and calculates her corresponding interim beliefs for each of the messages, $p(s^e)$; second, she computes the expected difference in value between the sources she is choosing between at each of the interim beliefs; and, finally, she takes a weighted average of those differences based on the forecasted probabilities calculated in the first step.[11] Although we focus on binary signals and two states, this perspective of taking weighted averages over posteriors is easily generalized to multiple states and more complex sources.

We continue the analysis by first calculating the difference in value across left and right biased sources for all possible priors, absent of an exogenous source. The red curve in Figure 2 below, shows the expected value of receiving a message from the right-biased source σ^R only at different prior beliefs (x-axis). The blue curve corresponds to the left-biased source σ^L.

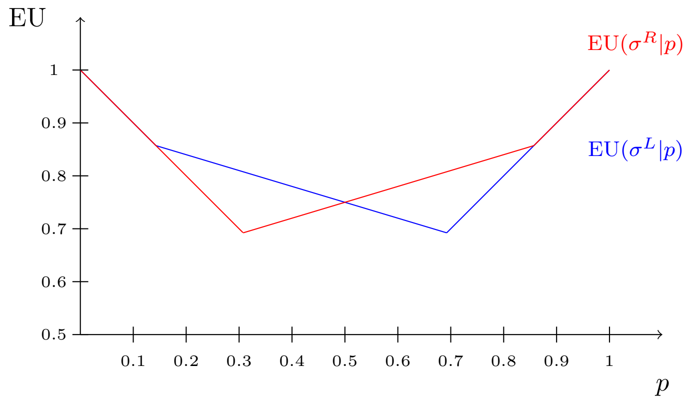

Figure 2: Expected utility of two symmetric signals, σ^L and σ^R

For both curves, there are three sections that correspond to different optimal actions. A DM who has a prior sufficiently close to 0 (is very certain that the state is L), finds it optimal to always choose A^L, no matter what message she receives. A DM who has a prior sufficiently close to 1, finds it optimal to always choose A^R, no matter what message she receives. Otherwise, the DM finds it optimal to match her action to the message (i.e. action A^L if the message is l and A^R if the message is r). The left-biased source is comparatively

[11]This framing is still valid even when the DM receives the message of her chosen source before the message of the exogenous source.

more accurate when the state is L. Therefore, if the DM finds L more likely according to her prior, she gets weakly greater expected value from choosing the left-biased source. Similarly if the DM believes R is more likely, she gets weakly greater expected value from the right-biased source. This goes along with previous literature showing that Bayesian agents find own-biased learning optimal, that is, signals which are biased towards the state that the agent finds more likely.

Next we see what happens when we add the expectation of additional information. Since the DM is Bayesian, the expectation of her interim posteriors must be her prior belief from an ex-ante perspective. Considering a binary exogenous source, this implies that one interim belief will be weakly to the left of the prior and one weakly to the right. The probability of going to either interim belief is proportional to the relative distance away from the prior. In the next figure, we plot the difference in value of the two available sources, $EU(\sigma^L|p) - EU(\sigma^R|p)$, for any given belief of the DM, p. Without the exogenous source, the DM's optimal choice is σ^L for all prior beliefs where the graph is above 0 and σ^R for all priors where the graph is below 0. One can see in the graph below that σ^L is weakly preferred for any belief below 0.5 and σ_R is weakly preferred for any belief above 0.5.

However, adding an exogenous source of information can reverse this optimal choice. Consider a DM who has a prior p_0 just below 0.5, as shown in the figure. Moreover, consider a binary exogenous source which causes the DM to have posterior $p(l^e)$ when the exogenous source sends the left message, l^e, and posterior $p(r^e)$ when the exogenous source sends the right message r^e. Since the DM is Bayesian, these posteriors must be weighted so that they are equal to the prior in expectation. To calculate the weighted average of the relative values of the signals at these posteriors, one can draw a line connecting the graph at each posterior. The weighted average then is the blue dot, where this line intersects the prior. This blue dot, represents the expected difference in value between the right and left biased sources, given the exogenous source. In this example, the blue dot is below 0, which implies that the right-biased source, σ_R, provides the DM with more value. This is the case, even though the DM found the left biased source more valuable absent the exogenous source. This is an example of how the expectation of additional information can possibly change the optimal choice of signal for the DM.

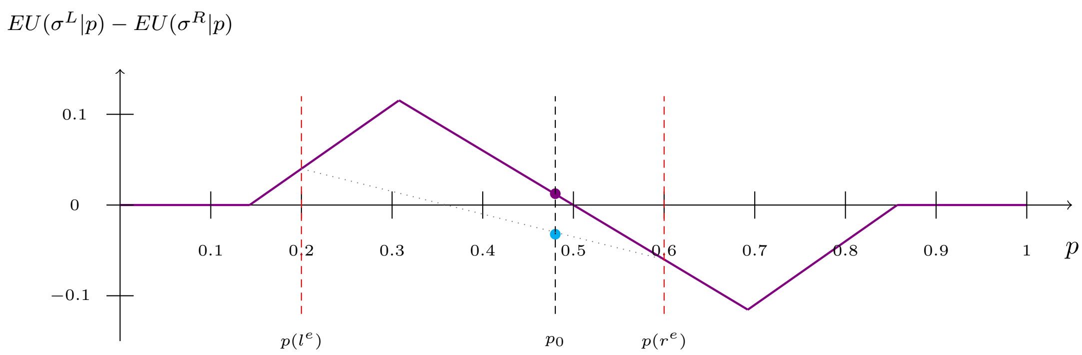

Figure 3: Example of the external signal changing the optimal signal choice

This example shows that it is possible for the exogenous source to change which signal is the most valuable. Using our framework, one can show this reversal of the optimal choice of information can happen in many situations, captured by the proposition below.

Proposition 1 *For any two sources,* σ^1, σ^2*:*
If there exists a belief, p*, s.t.* $Eu(\sigma^1|p) > Eu(\sigma^2|p)$*,*
then $\forall p_0 \in (0,1) : \exists \sigma_e, \text{ s.t.} : EU(\sigma^1, \sigma_e|p_0) > EU(\sigma^2, \sigma_e|p_0)$

Proof. Proof in Appendix. The proof is easily extendable to signals with finite and continuous message spaces as well as a more general action and state space, see note in appendix.

■

This shows that no matter what an agent's prior beliefs are, the expectation of additional information can always make the choice of one source optimal, as long as that source is optimal at least at one prior. This means that the exogenous source can change the news consumption decision for any non-trivial choice of news source.[12] This exemplifies the importance of including the exogenous source in any problem of information choice, since it can reverse the optimal choice for all possible prior beliefs.

[12]If a source is less valuable for all priors, no prior will ever find it optimal, and therefore the choice is trivial

In general, as one becomes more certain about what the state of the world is, ex-ante one expects to make more accurate decisions. Therefore, sources of information provide less value, and the difference between sources becomes less important. The extreme case is someone who is 100% certain of the state. These extreme priors find no value in any source, since they do not have any unresolved uncertainty. One can always find an exogenous source with a distribution of posteriors that put sufficient weight on posteriors that value more a specific source (σ^1), while placing the remaining weight on posteriors where the DM is indifferent between the sources she is choosing from, e.g. 0 or 1. This causes the DM to prefer that source (σ^1) on average. This is the intuition behind the proof of Proposition 1. These intuitions and this result do not rely on the binary nature of the the state space, message space nor action space. They are easily generalized to more general information choice settings with more complex action and state spaces, as well as more complex message structures.

We showed how the perspective from Observation 1 can be used to understand the importance of including the exogenous source in the analysis of information choice. We now continue with the analysis of our motivating example of optimal media-bias. More specifically, we investigate the impact of expecting to receive information in the future on the optimal choice of news bias.

5.1 Illustrative example – continuation

Before proceeding to a more general presentation of the results, we will use a concrete example to illustrate the different possible outcomes. Consider again Ann choosing among two symmetric news sources, σ^L and σ^R. But now we consider a more realistic setting where in both states the sources messages may not match the state. The probability that each of the sources sends message l or r is represented in Tables 2 and 3.

First, we will look at Ann's optimal choice when she does not expect to receive any additional information. In line with the previous literature, in that case she will weakly

State/Message	l^L	r^L
L	$\frac{4}{5}$	$\frac{1}{5}$
R	$\frac{1}{2}$	$\frac{1}{2}$

Table 2: Probability of messages from a left-biased source (σ^L)

State/Message	l^R	r^R
L	$\frac{1}{2}$	$\frac{1}{2}$
R	$\frac{1}{5}$	$\frac{4}{5}$

Table 3: Probability of messages from a right-biased source (σ^R)

prefer own-biased learning. More specifically, Ann will be indifferent between any news source if she is already very certain about one of the states, i.e. whenever her prior, p_0, is below $\frac{2}{7}$ or above $\frac{5}{7}$. That is because for such priors, no message from any of the available news sources could change her vote. For the remaining range of priors, if she attaches a higher probability to L (R) being the best alternative, she will find it optimal to choose σ^L (σ^R). This is illustrated in Figure 4.

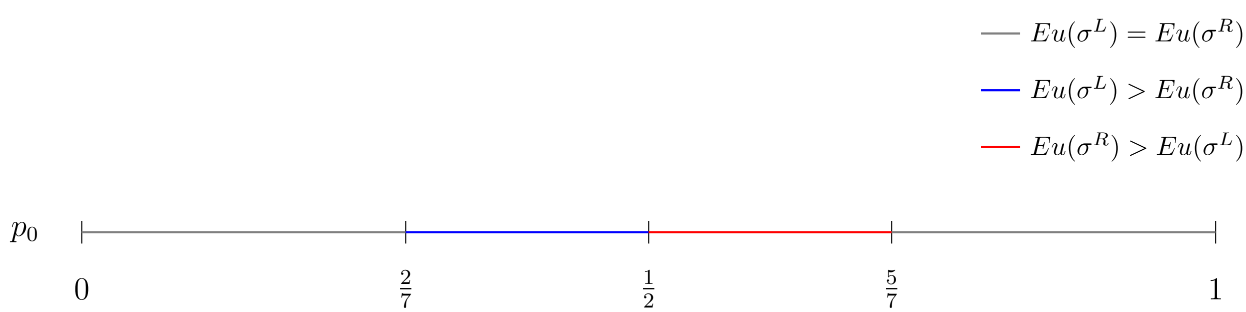

Figure 4: Optimal choice of signal in isolation

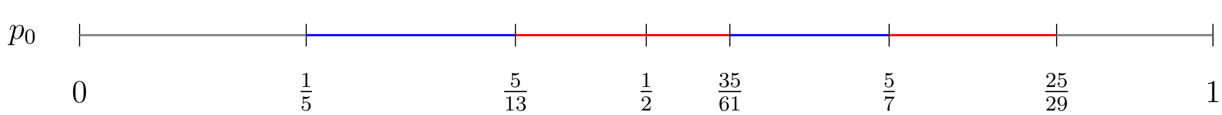

Figure 5: Optimal source choice, expecting an additional message from a right-biased source

Now suppose that when choosing a source Ann expects Bob to share the news he read with her (before she votes). Moreover, she expects Bob to read and, therefore, share right-biased news. As explained above, when choosing her news today she takes this expectation into account. In line with the first example and with Proposition 1, this may affect the news she reads. Indeed, her optimal choice (displayed in Figure 5) will depend on her prior belief in a very rich way. In particular, there are six non-symmetric adjacent intervals of priors that characterize her optimal choice of news source. Next, we will go over the different intervals to understand the intuition behind Ann's choice at each of them.

Recall from Observation 1 that Ann's problem could be looked at as follows. First, she considers what interim posterior belief each of Bob's messages would induce. Then, she compares the value of choosing each of the sources at such interim belief. And, finally, she weights those by the probability of ending up in each of them. We will see how this type of reasoning will be helpful to understand Ann's behavior at each region of priors.

Starting from the left-hand side, consider the first interval in Figure 5. For such priors, Ann is extremely certain that the state is L to the extent that she will vote L regardless of the information she receives. Since the information that she chooses has no effect on her vote and, thus, her payoff, she is indifferent between any news source. Going back to Observation 1, Ann's prior being below $\frac{1}{5}$ is equivalent to (both of) her interim posteriors being below $\frac{2}{7}$. Therefore, by the solution to Ann's problem when she learns in isolation, when Ann's prior is below $\frac{1}{5}$, she is indifferent between the two sources at both of her interim posteriors, leading to indifference also ex ante. An example of such a prior is represented in Figure 6.

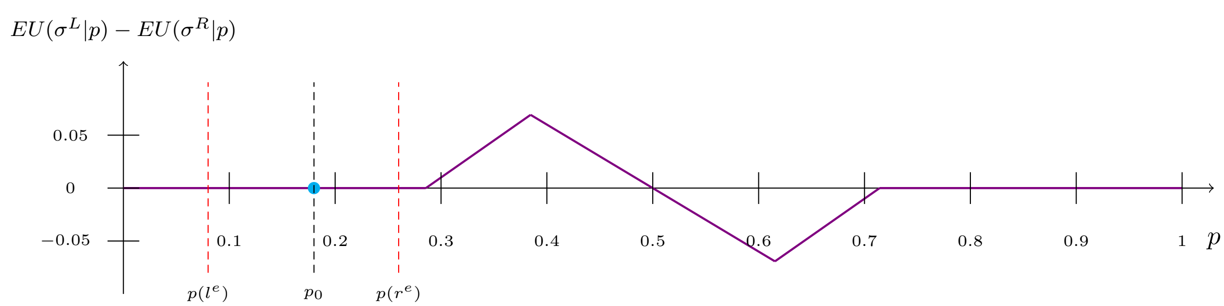

Figure 6: Ann's problem when she is extremely certain that the state is L

Consider now the second interval (in blue). At such priors, Ann is so certain that the best alternative is L, that all of her possible interim posteriors will be left-biased (below $\frac{1}{2}$). That is, a message from Bob alone would not be enough to change her vote. This was also true for the first interval. However, now there is some room for Ann to learn something valuable, in the sense that the aggregate information that she receives might change her vote. Therefore, she is no longer indifferent between sources. In particular, Ann's optimal choice will be to read the left-biased source (own-biased learning). The reason why, is that, since both of her interim posteriors will be left-biased, she will find the left-biased source weakly optimal at both of them. Moreover, since at (at least) one of them she will find the left-biased source strictly optimal, she strictly prefers own-biased learning ex ante. In fact, Ann's prior being above $\frac{1}{5}$ ensures that her interim posterior after a right message from Bob is above $\frac{2}{7}$, so that she is not indifferent between sources. On the other hand, her prior being below $\frac{5}{13}$ ensures that none of her interim posteriors is right biased (above $\frac{1}{2}$). Figure 7 displays an example of such a prior.

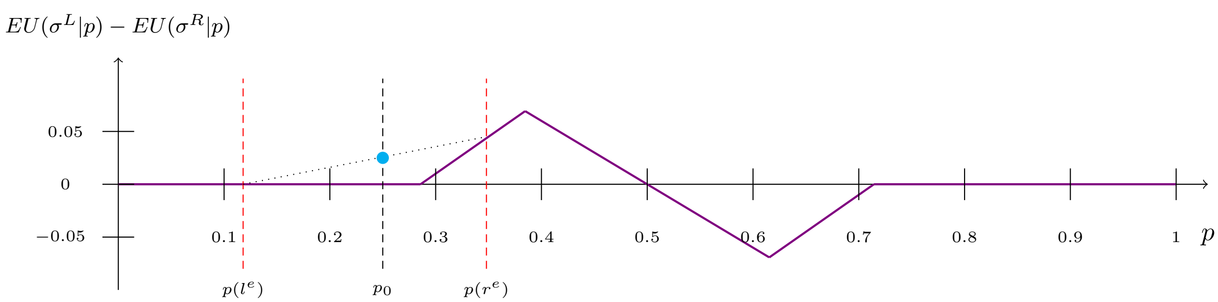

Figure 7: Ann's problem when she is very certain that the state is L

In the third interval from the left, Ann finds it optimal to choose the right-biased source (in red). Since the interval includes the neutral prior, $\frac{1}{2}$, this includes regions where Ann would be choosing own and opposite-biased learning. The intuition behind this result is that, at such priors, Ann is sufficiently uncertain about what to vote such that each of Bob's possible messages would result in her being biased in a different direction. As we saw before, this would mean that at each of her interim posteriors she would have a different preference for news bias. Which source ends up being optimal depends on the exact difference in value between sources at each interim posterior and the probabilities that they are realized.

When Ann's prior is slightly above $\frac{5}{13}$, Ann's interim posterior after Bob sends a left message is sufficiently low that she would be indifferent at such posterior. While her interim posterior after a right message would be slightly above $\frac{1}{2}$, making her strictly better off when choosing the right-biased source (opposite-biased learning). An example of this is shown in Figure 8. Another example where Ann would choose the right-biased source is plotted in Figure 9. In that case, Ann is pretty uncertain about the state but attaches slightly more probability to R being the best option and it is optimal for her to choose own-biased learning. In the language of Observation 1, she strictly prefers the left-biased source at her interim posterior after a left message and the right-biased source after a right message. But the second dominates when taking the weighted average.

The fourth interval from the left (in blue) follows the same intuition as the previous. When Ann is moderately certain that the best alternative is R, one of Bob messages ("right") will convince her more of her vote, while the other ("left") will change her view of what is the best alternative. As before, since each of her interim posteriors will have different biases, her optimal choice will be determined by the weighted average of the differences in the sources' value at each interim posterior. In this interval, the left interim posterior (after a left message) turns out to be more important. Thus, it is optimal for Ann to choose the left-biased source (opposite-biased learning).

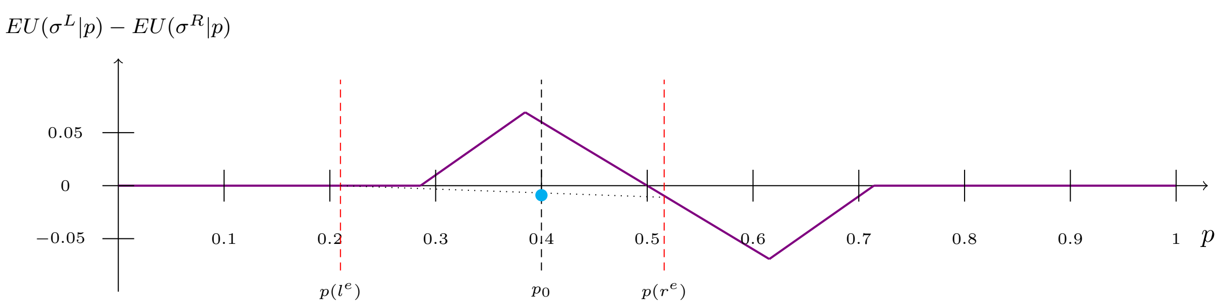

Figure 8: Ann's problem when she is moderately certain that the state is L

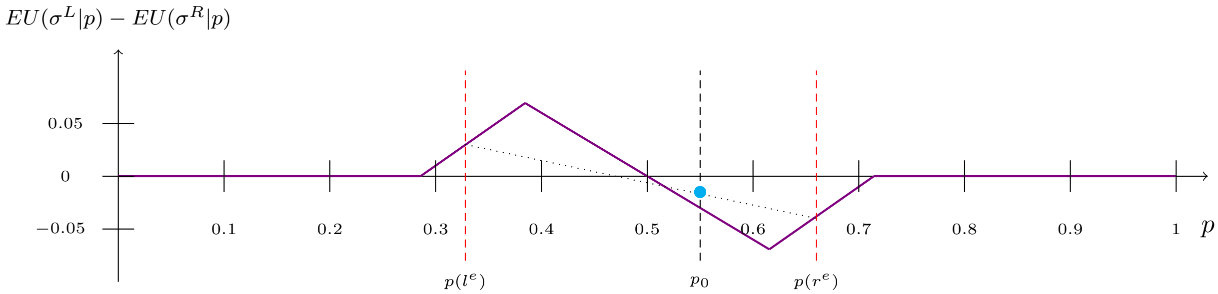

Figure 9: Ann's problem when she is relatively uncertain about the state

The fifth interval (in red) covers the priors at which Ann is very certain that the best alternative is R. Analogously to the second interval, she will find it optimal to choose own-biased learning. When Ann's priors are in this range, a left message from Bob would not be enough to change her vote. Thus, all her interim posteriors would be right-biased. This ensures that she weakly prefers the right-biased source. Since she is sufficiently uncertain about the state that two left messages can change her vote, the choice of news source will impact her expected utility. Therefore, her preference will be strict and her only optimal choice will be the right-biased source.

Finally, the last interval follows the same logic as the first one. When Ann is extremely sure that the best alternative is R so that no news can change her vote, she will be indifferent between the two sources. In the language of Observation 1, if Ann's prior is within this region, both of her interim posteriors will be above $\frac{5}{7}$. This implies that she would be indifferent at any of them and, thus, also ex ante.

The last remark from this example will be that certain features of the information that Ann expects to receive will affect her optimal strategy in an interesting and predictable way. An example of this is the bias of Bob's news. For instance, when Ann expects Bob to read (and thus share) left-biased news, her news choice will also be characterized by six non-symmetric adjacent intervals. This is displayed in Figure 10. As before, there are two extreme regions of indifference, two second-most extreme regions of own-biased learning and two regions in the middle where Ann finds either the left or the right biased source optimal. However, the exact thresholds of each interval are indeed affected by the bias of Bob's news. Note, for example, the difference in where the neutral prior $\frac{1}{2}$ lies: when Ann is fully uncertain about the best action to take and she expects Bob to share right-biased news, she finds it optimal to read right-biased news too, while she would find it optimal to read left-biased news if she expects Bob to share left-biased news too.

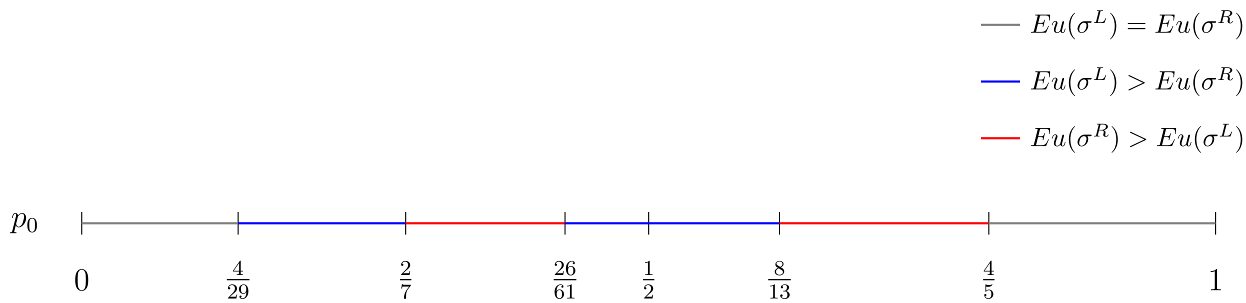

Figure 10: Optimal choice of signal expecting an additional left-biased message

This example illustrates how expectations of future information may change the optimal informational choices of an agent. Many features of Ann's optimal learning strategy will extend to other learning problems where a decision maker expects to receive information beyond her control in the future. More specifically, the pattern of the optimal strategy as well as the intuitions of this example generalize to a broad class of settings, which we now move on to.

5.2 The optimal strategy

The objective of this section is to understand how different features of expected future information affect the DM's optimal choice of news bias. From here on, we will assume that σ^L and σ^R are symmetric.[13] This is useful to shut down potentially confounding channels explaining the DM's choice, such as differences in overall informativeness between them. All

[13]Some of the results, as they are written now, do not accommodate sources that are fully informative about one the states, such as the ones in the example of Section 3. We are working on adapting the results to include them too. For now, the full characterization for such cases can be found in the application of Section 6.

proofs are relegated to the Appendix.

The first result of this section generalizes the optimal learning strategy discussed in the example above. Proposition 2 identifies a meaningful and interpretable sufficient condition under which the DM's optimal choice of source has the following structure. If the DM is extremely sure of the state, she is indifferent between any news source. If she is very sure of the state, she chooses own-biased learning. If she is moderately certain of the state, she chooses opposite biased learning. Lastly, if she is very uncertain about the state (p_0 close to $\frac{1}{2}$), she will choose the news that have the same bias as σ_e.

Proposition 2 *If* $\frac{\pi^L(r|R)^2}{\pi^L(r|L)^2} \geq \frac{\pi^e(l|L)}{\pi^e(l|R)}\frac{\pi^e(r|R)}{\pi^e(r|L)} \geq \frac{\pi^L(l|L)}{\pi^L(l|R)}\frac{\pi^L(r|R)}{\pi^L(r|L)}$, *then, there exist five thresholds* $0 < p_1 < p_2 < p_3 < p_4 < p_5 < 1$ *s.t.*

i) If $p_0 \in [0, p_1] \cup [p_5, 1]$, *the DM is indifferent between any choice of* σ_i.

ii) If $p_0 \in [p_1, p_2] \cup [p_4, p_5]$, *the DM finds own-biased learning optimal.*

iii) If $p_0 \in [p_2, \min\{p_3, \frac{1}{2}\}] \cup [\max\{p_3, \frac{1}{2}\}, p_4]$, *the DM finds opposite-biased learning optimal.*

iv) If $p_0 \in [\min\{p_3, \frac{1}{2}\}, \max\{p_3, \frac{1}{2}\}]$, *the DM's optimal choice is* $\sigma_i = \sigma_e$.

Before discussing the intuition behind the optimal choice in each region of priors, we will explain the meaning of the sufficient condition. First, consider the ratio $\frac{\pi^x(l|L)}{\pi^x(l|R)}$ for a source σ^x. This ratio summarizes the informativeness of message l from σ^x about the state being L. The larger this ratio is, the more a DM would update her prior in the left direction if she observes σ^x sending message l. Analogously, the greater $\frac{\pi^x(r|R)}{\pi^x(r|L)}$, the more informative is the message r from σ^x. The following lemma builds on this intuition and will be useful in interpreting the meaning of the condition.

Lemma 1 *A signal* σ^x *is left-biased iff* $\frac{\pi^x(r|R)}{\pi^x(r|L)} > \frac{\pi^x(l|L)}{\pi^x(l|R)}$, *right-biased iff* $\frac{\pi^x(l|L)}{\pi^x(l|R)} > \frac{\pi^x(r|R)}{\pi^x(r|L)}$ *and, otherwise, it is unbiased.*

By lemma 1, a left-biased source is more informative when sending a message r than l and a right-biased source is more informative when sending a message l than r. The intuition is that the source sends such message less often, but, when it does it, the probability that the message is truthful is larger. Another implication of this lemma is that when σ^L and σ^R are symmetric, the left-biased source will be equally informative when sending message r (l) as the right-biased source when sending message l (r). In addition, note that for any σ^x, $\frac{\pi^x(l|L)}{\pi^x(l|R)} > 1$.

Next, consider the product of ratios $\frac{\pi^x(l|L)}{\pi^x(l|R)}\frac{\pi^x(r|R)}{\pi^x(r|L)}$ for a source σ^x. This can be interpreted as a measure that captures overall informativeness, since each separate ratio increases with the source's informativeness about a specific state. To make this point clearer, we plot some isocurves of sources with the same level of overall informativeness, that is, the same value of $\frac{\pi^x(l|L)}{\pi^x(l|R)}\frac{\pi^x(r|R)}{\pi^x(r|L)}$ in Figure 11. The dotted white triangle in Figure 11 represents the space of all sources (binary signals), where the x-axis is the probability of sending message l when the state is L and the y-axis is the probability of sending message r when the state is R. The sources in the dotted line that goes from $(0, 1)$ to $(1, 0)$ are all the uninformative signals and the point $(1, 1)$ represents the fully informative source. Recall that we assumed both of them away. The vertical and horizontal dotted lines correspond to the sources that are fully informative in one of the states (like the ones in the first illustrative example and in our application). Sources in the gray area are redundant, because they are equivalent to some other source in the white triangle up to interchanging the labels l and r.

Each of the plotted curves corresponds to a certain level of overall informativeness, that is, a certain value of $\frac{\pi^x(l|L)}{\pi^x(l|R)}\frac{\pi^x(r|R)}{\pi^x(r|L)}$. Movements to the right and up correspond to greater levels of overall informativeness. Note that an increase in informativeness can come from a greater probability of the source's messages being correct at one or both states, e.g. moving from C to B. But also from a rise in biasedness, that is, an increase in the difference between the probability of sending the correct message in one state versus the other, e.g. the movement from B to C. The intuition behind this is that, when moving from C to B there is a larger probability overall that the source sends the correct message (a message that matches the state), while when moving from B to A, although this probability is lower, this is compensated by the rise in informativeness of one of the messages, i.e. r.

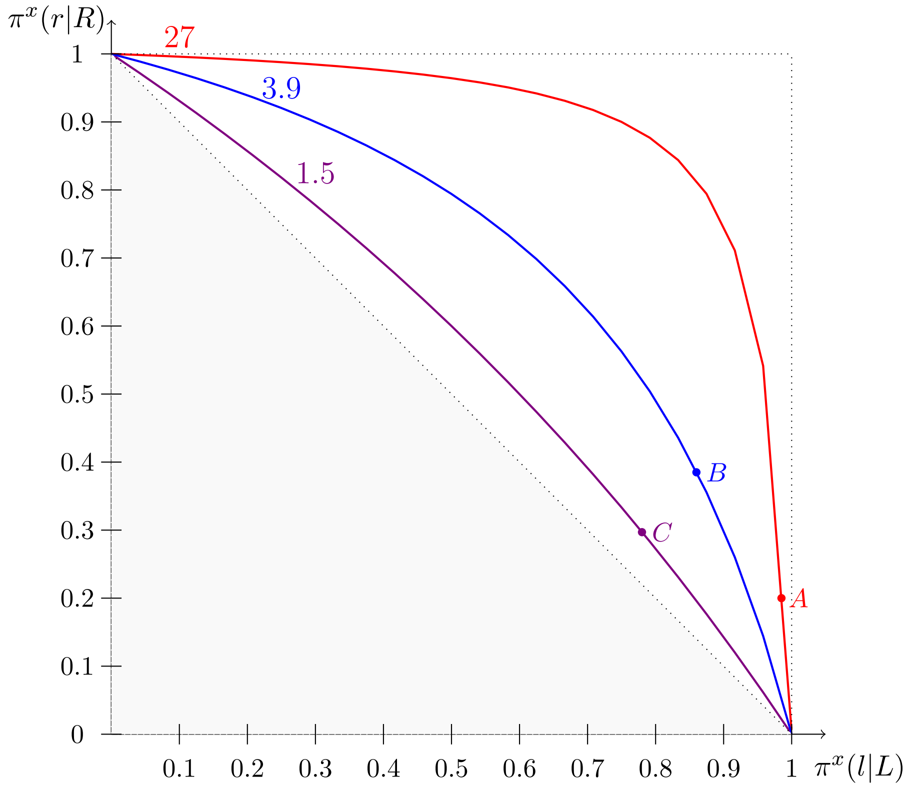

Figure 11: Isocurves for $\frac{\pi^x(l|L)}{\pi^x(l|R)}\frac{\pi^x(r|R)}{\pi^x(r|L)} = 27$; $\frac{\pi^x(l|L)}{\pi^x(l|R)}\frac{\pi^x(r|R)}{\pi^x(r|L)} = 3.86$; $\frac{\pi^x(l|L)}{\pi^x(l|R)}\frac{\pi^x(r|R)}{\pi^x(r|L)} = 1.5$

By Lemma 1, it should always be the case that $\frac{\pi^L(r|R)^2}{\pi^L(r|L)^2} > \frac{\pi^L(l|L)}{\pi^L(l|R)}\frac{\pi^L(r|R)}{\pi^L(r|L)}$. Thus, given a specific pair (σ^L, σ^R), the set of sources that satisfy the sufficient condition in Proposition 2 will be non-empty. In addition, note that the natural case where $\sigma^e \in \{\sigma^L, \sigma^R\}$ is always covered by this proposition. An example for a given σ^L and σ^R is plotted in Figure 12, where the sources that satisfy the sufficient condition are colored in violet. The two curves represent the upper and lower bounds of the condition. The specific pair of (symmetric) σ^L and σ^R are also plotted in the figure.

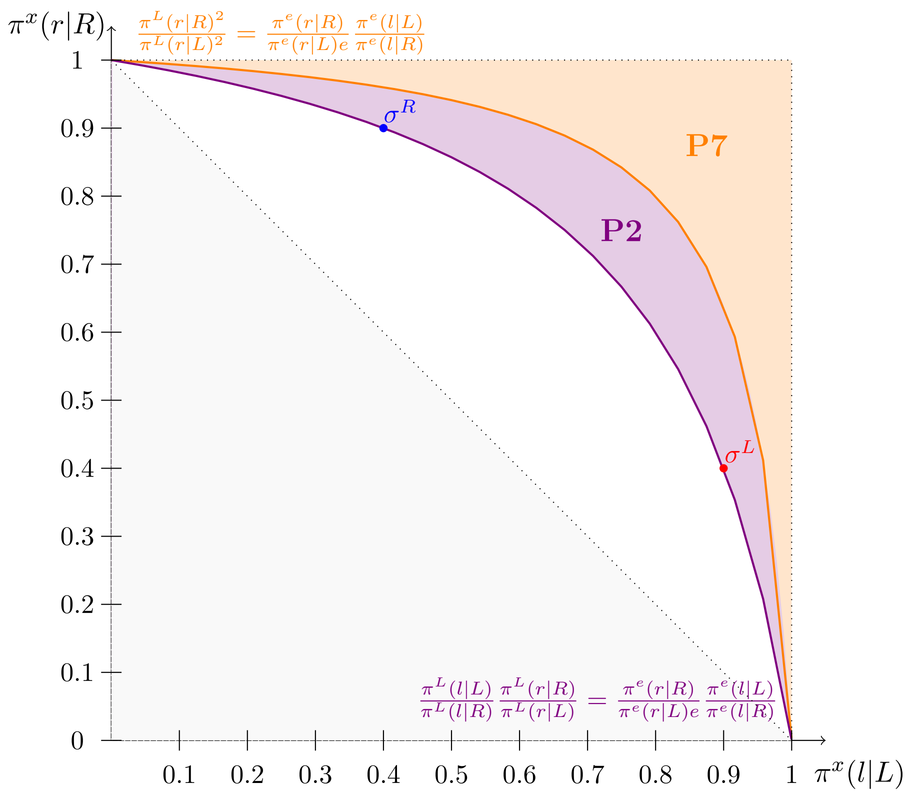

Figure 12: Scope of Propositions 2 and 7 for $\pi^L(l|L) = 0.9,\ \pi^L(r|R) = 0.4$

All in all, Proposition 2 establishes that, as long as σ^e is equally or more informative (up to some upper bound) than the sources available to the DM, the optimal learning strategy will have the same structure as in the example above. At the end of this section we will briefly discuss how the DM's optimal strategy looks like when the exogenous source is much more informative than the sources available to the DM (see Proposition 7).

Now we move on to discussing the different regions of priors in the characterization. Some of them will also arise when the DM expects other types of additional information, not covered by Proposition 2. Therefore, we will also provide results pointing this out. We will break our analysis of the DM's optimal strategy in different levels of certainty of the DM about the state of the world (different regions of priors which are further or closer to $\frac{1}{2}$).

Consider first a DM who is quite certain about the state of the world. In that case, there are three main *heuristics* that she may use:

No news can change my action– This is the case of the most extreme regions, that is, where the DM is the most sure about the state. When the DM is so sure that no combination of news (no combination of messages of both the exogenous and chosen source) can change her payoff relevant action, she will be indifferent between any news source. In the language of Observation 1 this corresponds to situations where the DM would be indifferent at both interim posteriors. Graphically, when plotting the difference in value between σ^L and σ^R, both interim posteriors would lie in one of the flat regions in the extremes. See an example of this in Figure 6.

Exogenous information cannot change my bias, but news are valuable– When the DM is very sure of the state but still can learn something valuable from the combination of (chosen and exogenous) news, she is no longer indifferent. In this case, if she is so certain of the state that exogenous information alone cannot change her bias, she will find it optimal to choose own-biased learning. By exogenous information not changing her bias we mean that her interim posteriors after any message of σ_e have the same bias as her prior. The intuition behind this is that, at both interim posteriors, it will be weakly optimal to choose own-biased learning and at, at least, one of them the preference will be strict. Figure 7 offers an example of such situation.

Exogenous information can change my bias, but makes me very uncertain– The idea here is that a message from σ^e can change the DM's bias, namely, one of her interim posteriors will have a different bias that her prior. But she is so certain of a message opposing her bias leaves her quite uncertain (the interim posterior is close to $\frac{1}{2}$), while the a message that confirms her bias would make her very certain of the state. In this case, she has opposing preferences at each interim posterior, since they lie in different sides of $\frac{1}{2}$. But since learning will be more valuable at the more uncertain interim posterior, she will find it optimal to choose opposite-biased learning. Figure 8 showcases this.

The next proposition summarizes the optimal strategy of a relatively certain DM for different levels of certainty. The different cases identified in the proposition naturally correspond to the different heuristics described above.

Proposition 3 *For any σ_e, there exist cutoffs $0 \le p_1 < p_2 \le p_3 < \frac{1}{2} < p_4 \le p_5 < p_6 \le 1$ such that:*

i) If $p_0 \in [0, p_1] \cup [p_6, 1]$, the DM is indifferent between any choice of σ_i.

ii) If $p_0 \in (p_1, p_2) \cup (p_5, p_6)$, the DM's optimal choice is own-biased learning.

iii) If $p_0 \in (p_2, p_3) \cup (p_4, p_5)$, the DM's optimal choice is opposite-biased learning.

If $\frac{\pi^L(r|R)}{\pi^L(r|L)} < \frac{\pi^e(l|L)}{\pi^e(l|R)} \frac{\pi^e(r|R)}{\pi^e(r|L)}$, then, $p_2 < p_3$ and $p_4 < p_5$.

Note that p_1 and p_2 in this proposition coincide with the two lowest thresholds in Proposition 2 and p_5 and p_6 with the two highest.

The most extreme region of priors, $[0, p_1] \cup [p_6, 1]$, corresponds to the first heuristic. This region always contains the extreme points 0 and 1, since there is no signal that can change the DM's belief when she is fully certain of the state. However, it will also contain other beliefs when sources are not fully informative in any state. The next observation provides a more formal account for this.

Observation 2 $p_1 > 0 \iff \max\{\pi^L(l|L), \pi^e(l|L)\} < 1$ *and* $p_6 < 1 \iff \max\{\pi^R(r|R), \pi^e(r|R)\} < 1$.

For instance, the setting in the example of Section 3 and the application in Section 6 is such that only a fully certain DM will find no bundle of sources valuable. Thus, $p_1 = 0$ and $p_6 = 1$. Instead, the setting in the example of Section 5.1. is such that no source is fully informative of any state. In that case, it is possible that a DM is not fully certain but still no bundle of sources can change her vote, so that $p_1 > 0$ and $p_6 < 1$.

Another interesting observation is that, as reflected by the heuristic, p_1 coincides with the prior at which the right-most final posterior, $p(r^e, r^L)$, equals $\frac{1}{2}$. By the same logic, p_6 is the DM's prior at which the left-most final posterior, $p(l^e, l^r)$, does. The idea is that if $p(r^e, r^L)$ is just above $\frac{1}{2}$, consuming the left-biased source can generate some valuable learning (by changing the DM's payoff relevant action from A^L to A^R), while the right-biased source cannot. Therefore, the DM would be no longer indifferent at a prior just above p_1. An analogous logic works for p_6.

The second most extreme region of priors, $(p_1, p_2) \cup (p_5, p_6)$, corresponds to the second heuristic. Note that this region is never empty. Thus, there is always some prior at which the DM finds it optimal to choose own-biased learning. This region contains (at least) all priors for which: i) both interim posteriors have the same bias as the prior belief, and ii) at (at least) one of them learning is valuable.

In contrast, whether the third heuristic is used by the DM at some prior, might depend on the structure of the exogenous information. Proposition 3 identifies a sufficient condition for this heuristic to be used. The condition requires that the informativeness of the exogenous source is above a certain lower bound, which increases with the informativeness of σ^L and σ^R. This can be interpreted as follows. If the additional information is sufficiently good, there will exist some range of moderately certain priors that will find opposite-biased learning optimal. Figure 13 illustrates this lower bound for a specific σ^L and σ^R. As exemplified there, even if the condition is only sufficient, the scope of this region tends to be big. Note that the settings where Proposition 2 holds are always included in this region. Formally, this is due to the fact that the informativeness ratios for each message are always strictly larger than 1. Thus, the lower bound in Proposition 2's sufficient condition is always above the lower bound in Proposition 3. Namely, $\frac{\pi^L(r|R)}{\pi^L(r|L)} < \frac{\pi^L(r|R)}{\pi^L(r|L)} \frac{\pi^L(l|L)}{\pi^L(l|R)}$.

Before analyzing the optimal strategy of a more central DM, we will remark why the third heuristic is sensible to explain the opposite-biased learning region in Proposition 3. The following lemmas will be helpful in explaining why.

Lemma 2 *Given that* $\frac{\pi^L(r|R)}{\pi^L(r|L)} < \frac{\pi^e(l|L)}{\pi^e(l|R)} \frac{\pi^e(r|R)}{\pi^e(r|L)}$*, there exist* $\epsilon, \eta > 0$ *such that if* $p_0 \in N_\epsilon(p_2)$*, then,* $EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e)) = 0$ *and* $p(r^e) \in N_\eta(\frac{1}{2})$*. In addition, if* $p_0 = p_2$*, then,* $p(r^e) = \frac{1}{2}$.

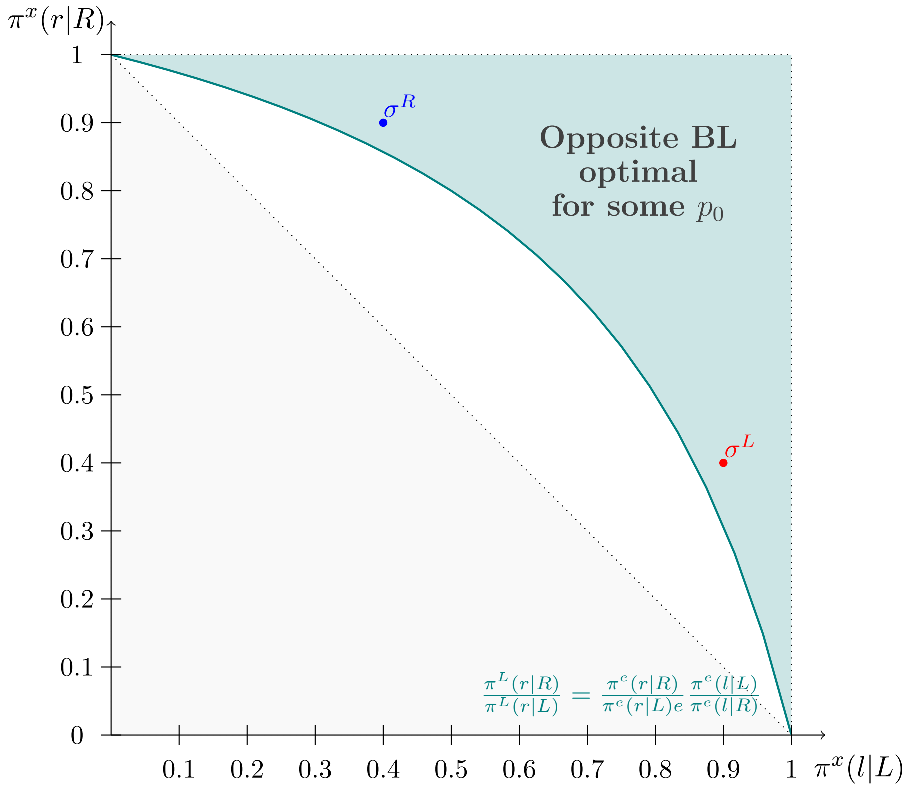

Figure 13: Sufficient condition in Proposition 3 for $\pi^L(l|L) = 0.9, \ \pi^L(r|R) = 0.4$

Lemma 3 *Given that* $\frac{\pi^L(r|R)}{\pi^L(r|L)} < \frac{\pi^e(l|L)}{\pi^e(l|R)} \frac{\pi^e(r|R)}{\pi^e(r|L)}$*, there exist* $\epsilon, \eta > 0$ *such that if* $p_0 \in N_\epsilon(p_6)$*, then,* $EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e)) = 0$ *and* $p(l^e) \in N_\eta(\frac{1}{2})$*. In addition, if* $p_0 = p_6$*, then,* $p(l^e) = \frac{1}{2}$.

The first part of Lemma 2 is saying that, under the sufficient condition in Proposition 3, when the DM's prior is sufficiently close to the threshold p_2, her left interim posterior will be so low that she will be indifferent between sources, while the right one will be close to $\frac{1}{2}$. Putting this together with the second part, if the DM's prior is right above p_2 (still in its neighborhood), then, it should be that: i) $p(r^e)$ is right above $\frac{1}{2}$ and ii) the DM is indifferent between sources at $p(l^e)$. Since at an interim posterior right above $\frac{1}{2}$ the DM will strictly prefer the right-biased source, a DM with a prior slightly above p_2 should strictly prefer the right-biased source. By continuity, this reasoning generates an interval of priors right above p_2 where opposite-biased learning is optimal. The example in Figure 8 follows exactly this logic. Similarly, by Lemma 3, under the sufficient condition in Proposition 3, when the DM's prior is right below p_6, she will be indifferent at the right interim posterior, because she will be sufficiently certain. On the other hand, she will strictly prefer the left-biased source at the left interim posterior, since it is right below $\frac{1}{2}$. Therefore, opposite biased learning will be optimal.

Next, we will analyze the problem of a relatively uncertain DM. More specifically, we will analyze the optimal choice of a DM whose prior is in the neighborhood of $\frac{1}{2}$. The following three propositions will provide a full characterization (covering all the space of binary exogenous sources) of the optimal strategy for such a DM. Figure 16 illustrates the space of sources covered by each proposition for a specific σ^L and σ^R. Again, there are three main heuristics that a DM who is relatively uncertain about the state of the world may use:

Exogenous information is so powerful that my news choice is irrelevant– When the exogenous source is very informative about both states, the DM's choice of news source becomes irrelevant. In the sense that no matter what news source she chooses, she will end up choosing the (payoff relevant) action that matches the message sent by the exogenous source. Therefore, a relatively uncertain DM who expects to receive additional news that are very informative about both states, will be indifferent between sources. Figure 14 illustrates this.

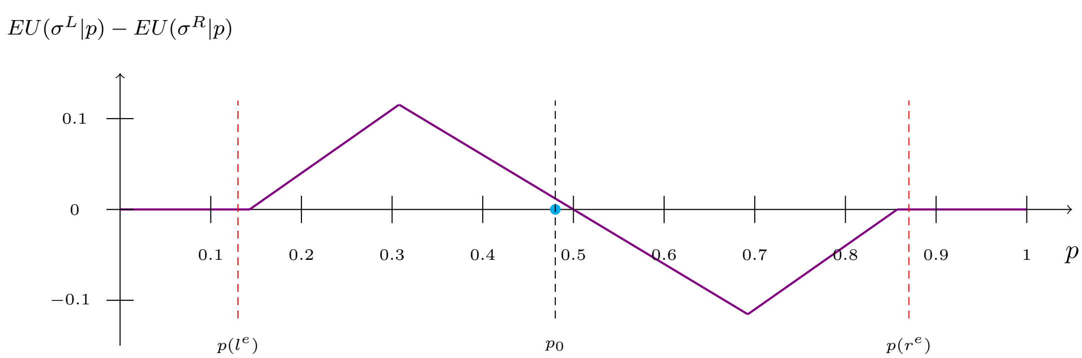

Figure 14: An exogenous source that makes the DM indifferent

Exogenous information is so weak that I follow the message of my news– If the exogenous source is not very informative about any of the states, the DM's choice of news will be "as if" she expected no additional information. When a DM is relatively uncertain (her prior is around $\frac{1}{2}$) and only receives a message from one source, she will choose the action that agrees with the message sent. And if the exogenous information is not very powerful, a relatively uncertain DM will still be relatively uncertain at any of her interim posteriors (after each message of the exogenous source). Therefore, she would choose the (payoff relevant) action that agrees with the message sent by her chosen news source, regardless of σ^e's message. In that case, her problem looks "as if" she expected no additional information and, thus, chooses own-biased learning. Figure 15 displays an example of this. Indeed, one can see that the blue dot, which represents the DM's expected value of the news bundle (σ^L, σ^e) versus (σ^R, σ^e), lies exactly at the DM's expected value of the news source σ^L versus σ^R alone, "as if" σ^e was not present.

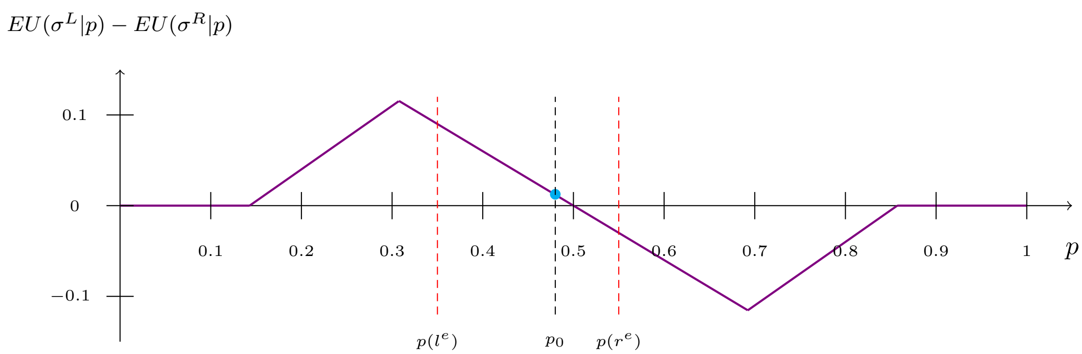

Figure 15: An exogenous source that has no effect on the optimal choice

The DM matches the bias of the exogenous source– When the exogenous source and the news sources available to the DM are comparably informative, the exogenous source's bias will determine at what interim posterior the DM finds information more valuable. For a relatively uncertain DM, the left interim posterior will be further from her prior than the right interim posterior if and only if the exogenous source is right-biased. This also means that the left interim posterior will be further from $\frac{1}{2}$. Thus, the DM will be more certain at the left interim posterior than the right one. In addition, the right interim posterior will naturally be put more weight, because it is realized with a larger probability (given that the exogenous source is right-biased). Since information will be more valuable at the right interim posterior, this will be given more weight. Thus, the preference for a right-biased source will dominate. An analogous argument works if the exogenous source is left-biased.

Proposition 4 describes the optimal decision of an uncertain DM when the exogenous source is much more informative than the sources available to the DM, namely, $\min\left\{\frac{\pi^e(r|R)}{\pi^e(r|L)}, \frac{\pi^e(l|L)}{\pi^e(l|R)}\right\} > \frac{\pi^L(r|R)}{\pi^L(r|L)}$. In that case, the interim posteriors of the DM will be so certain, that no information she can choose will be valuable. Therefore, following the first heuristic above, she will be indifferent between sources.

Proposition 4 *If* $\min\left\{\frac{\pi^e(r|R)}{\pi^e(r|L)}, \frac{\pi^e(l|L)}{\pi^e(l|R)}\right\} > \frac{\pi^L(r|R)}{\pi^L(r|L)}$*, then, there exists an interval* $[\underline{p}, \bar{p}]$*, such that* $\frac{1}{2} \in [\underline{p}, \bar{p}]$ *and, if* $p_0 \in [\underline{p}, \bar{p}]$*, the DM is indifferent between sources.*

On the other extreme, Proposition 5 establishes that, whenever the exogenous source is not very informative compared to the sources the DM can choose from, $\frac{\pi^L(l|L)}{\pi^L(l|R)} > \max\left\{\frac{\pi^e(r|R)}{\pi^e(r|L)}, \frac{\pi^e(l|L)}{\pi^e(l|R)}\right\}$, a relatively uncertain DM will find own-biased learning optimal. In accordance with the second heuristic above, under this condition, the additional information is not powerful enough to change the DM's preferences.

Proposition 5 *If* $\frac{\pi^L(l|L)}{\pi^L(l|R)} > \max\left\{\frac{\pi^e(r|R)}{\pi^e(r|L)}, \frac{\pi^e(l|L)}{\pi^e(l|R)}\right\}$*, then, there exists an interval* $[\underline{p}, \bar{p}]$ *such that* $\frac{1}{2} \in [\underline{p}, \bar{p}]$ *and, if* $p_0 \in [\underline{p}, \bar{p}]$ *the DM's optimal choice is own-biased learning.*[14]

Lastly, Proposition 6 covers the more realistic case, where the exogenous source is neither much more nor much less informative than the sources available to the DM. In such situations, an uncertain DM will find it optimal to choose the source that is biased in the same direction as the exogenous source. The intuition for this is in line with the third heuristic. The interim posteriors of a central DM will move less further from the prior in the direction of the source bias than in the opposite direction. Learning will be more valuable at the interim belief that is more uncertain (closer to $\frac{1}{2}$) and it will have more weight when computing the weighted

[14]For $p_0 = \frac{1}{2}$ own-biased learning is not a well-defined concept and, correspondingly, the DM is indifferent between any choice of σ_i.

average across interim posteriors. Therefore, the preference at such interim posterior will dominate. As explained above, the more uncertain interim posterior will be biased in the same direction as the exogenous source, thus, choosing news with the same bias as the exogenous source will be optimal. Figure 3 and 9 are both examples of this.

Proposition 6 *If* $\max\left\{\frac{\pi^e(r|R)}{\pi^e(r|L)}, \frac{\pi^e(l|L)}{\pi^e(l|R)}\right\} > \frac{\pi^L(l|L)}{\pi^L(l|R)}$ *and* $\frac{\pi^L(r|R)}{\pi^L(r|L)} > \min\left\{\frac{\pi^e(r|R)}{\pi^e(r|L)}, \frac{\pi^e(l|L)}{\pi^e(l|R)}\right\}$*, then, there exists an interval* $[\underline{p}, \bar{p}]$ *such that* $\frac{1}{2} \in [\underline{p}, \bar{p}]$ *and, if* $p_0 \in [\underline{p}, \bar{p}]$ *the DM finds it optimal to choose the source that is biased in the same direction as the exogenous source.*

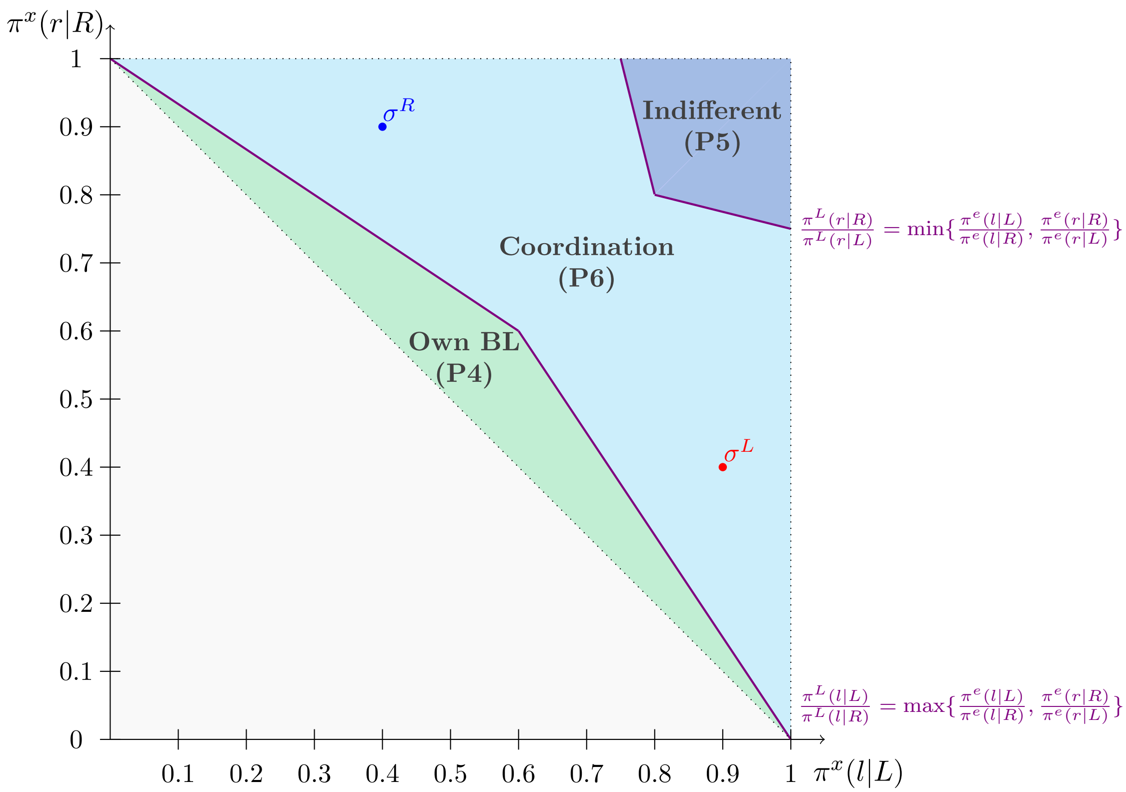

Figure 16: Optimal choice of an uncertain DM for $\pi^L(l|L) = 0.9, \;\; \pi^L(r|R) = 0.4$

We will close this section by characterizing the DM's optimal strategy, when the sufficient condition in Proposition 2 fails. In particular, when the news she expects to receive are much more informative than the ones she can access. Proposition 7 provides this characterization, which, as expected, contains the stylized features discussed above: i) the DM is indifferent in the most extreme regions of priors, ii) she chooses own-biased learning in the two second most extreme, iii) she chooses opposite-biased learning when moving further to the center, and, iii) she will either be indifferent or choose the source that has the same bias as the additional information when being very uncertain. This will depend on the specific structure of σ_e, σ^R and σ^L, as we saw in the analysis above.

Proposition 7 *If* $\frac{\pi^e(l|L)}{\pi^e(l|R)}\frac{\pi^e(r|R)}{\pi^e(r|L)} > \frac{\pi^L(r|R)^2}{\pi^L(r|L)^2}$*, then, there exist six thresholds* $0 < p_1 < p_2 < p_3 < p_4 < p_5 < p_6 < 1$ *s.t.*

i) If $p_0 \in [0, p_1]$*,* $p_0 \in [p_6, 1]$ *or* $p_0 \in [p_3, p_4]$*, the DM is indifferent between any* σ_i*.*

ii) If $p_0 \in [p_1, p_2]$ *or* $p_0 \in [p_4, p_5]$*, the DM's optimal choice is* σ^L*.*

iii) If $p_0 \in [p_2, p_3]$ *or* $p_0 \in [p_5, p_6]$*, the DM's optimal choice is* σ^R*.*

An example of the sufficient condition in Proposition 7 is plotted in Figure 12. A more detailed discussion about the features of this strategy is pending. In addition, in order to obtain a full characterization of the DM optimal choice of source for any binary exogenous source and for any prior belief it is left to solve for the case where $\frac{\pi^L(l|L)}{\pi^L(l|R)}\frac{\pi^L(r|R)}{\pi^L(r|L)} \geq \frac{\pi^e(l|L)}{\pi^e(l|R)}\frac{\pi^e(r|R)}{\pi^e(r|L)}$. We are currently working on this for completeness, but the above discussion provides many insights on the structure that it will have.

In the next section, we offer a meaningful application of our framework to a problem of social interaction.

6 Application to News Sharing

In the previous section, the additional information was exogenous. However, in many real life scenarios, the expected information is chosen by someone else. Therefore, in many situations, the complete set of information received by an agent is the outcome of some strategic process. Although the agent may not have direct control of all the information she receives, she may still indirectly affect the information chosen by others and shared with her through strategic interactions. For example, when choosing the bias of their news, people are aware that they will later share the news they read with other people and others will share what they read with them. In this case, the people they will interact with will provide them with news with a bias not under their direct control. At the same time, the other people also choose their news bias, knowing who they will interact with. In these situations the additional future information that each agent expects to receive is the outcome of some strategic equilibrium between all the people sharing news with one another. We will model this behaviour more closely in the remainder of this section.

6.1 Media-Bias Choice within a Group

We start by considering two agents who know the prior of the other player. Since there are now two players we denote the initial belief on the probability of state R of player i as p_i. The players have to choose what news source to read without being able to observe the other player's choice at the time of the medium choice. Concretely, the timing of the interaction is the following: in the first period both players choose a source without being able to observe the choice of the other player; in the second period the agents observe the messages of the chosen sources and share them with the other agent; finally, the agents take some action attempting to match the state just as in Section 3 above. Either agent can pick between the left-biased source and the right-biased source. As in our baseline model, we assume that the messages read by each agent are independent. The simultaneous game of choosing a source is illustrated in the matrix below.

	σ^L	σ^R
σ^L	$EU_i(\sigma^L, \sigma^L)$	$EU_i(\sigma^L, \sigma^R)$
σ^R	$EU_i(\sigma^R, \sigma^L)$	$EU_i(\sigma^R, \sigma^R)$

Matrix of payoffs of the two-by-two game for i

Since the choice of news bias is the topic of interest of this section, we will restrict the structure of the sources available to the agents. As in other relevant papers studying related issues, such as Che and Mierendorff [2019] and Gans [2023], we will assume symmetry of the sources and focus on the most extreme type of bias.[15] In particular, we will make the following assumption.

Assumption 1 *The agents choose between two symmetric sources, σ^L and σ^R, such that $\pi^L(l|L) = \pi^R(r|R) = 1$ and $\pi^L(r|L) = \pi^R(l|R) = \alpha$.*

In what follows, we will call α the quality of the information sources. It can be viewed as a general indicator of accuracy or trustworthiness in the information published in the media. When $\alpha = 0$, the source does not transmit any information with its messages, that is, it is uninformative. And as α increases the source becomes more informative of the state. We will denote the bias of agent i by $b_i \in \{L, R\}$, where $b_i = L$ means that i is left-biased (believes that the state is more likely L than R). Analogously, $b_i = R$ means that i is right-biased (believes that the state is more likely R than L). For completeness we will denote the opposite bias as $\bar{b}_i \in \{L, R\}$. For instance, if i is left-biased, $\bar{b}_i = R$.

In order to find the equilibrium of the game it is necessary to determine the agent's valuation for different bundles of news sources. The following proposition shows the ordering of the payoffs for the agents.

Proposition 8 *There exists $\hat{p} \in (0, \frac{1}{2})$ s.t.*

i) For players with $p_i \in [\hat{p}, 1 - \hat{p}]$,

$$EU_i(\sigma^{b_i}, \sigma^{b_i}) > EU_i(\sigma^{\bar{b}_i}, \sigma^{\bar{b}_i}) > EU_i(\sigma^{b_i}, \sigma^{\bar{b}_i})$$

[15]The reason why we focus on this type of bias is that, fixing a level of overall accuracy, it would be the most preferred bias.

ii) For players with $p_i \notin [\hat{p}, 1-\hat{p}]$,

$$EU_i(\sigma^{b_i}, \sigma^{b_i}) > EU_i(\sigma^{b_i}, \sigma^{\bar{b}_i}) > EU_i(\sigma^{\bar{b}_i}, \sigma^{\bar{b}_i}).$$

This shows that the relevant factor for determining an agent's strategy is how certain the players are about the state. For players in a central interval of priors characterized by $\hat{p}$ above, the payoffs are such that they wish to choose the same source as the other agent. Agents that are sufficiently certain, with priors outside of the central interval, have a dominant strategy to choose a source that is biased in the direction of their prior (own-biased learning).

Proposition 9 $\hat{p} = \frac{1-\alpha}{2-\alpha}$ *continuously decreases with the quality of the news sources,* α*.*

This proposition shows that the size of the interval of uncertain players, who prefer to coordinate their news source with the other agent, depends on the informativeness of the sources. As the sources become more informative, more priors find it optimal to coordinate instead of choosing a source biased in the direction of their initial prior. We can now use the agents best-responses to characterize the Nash equilibria of this sequential game.

Proposition 10 *There exists a unique Nash Equilibrium iff at least one player's prior is outside of* $[\hat{p}, 1-\hat{p}]$*:*

i) If both players' prior is outside of $[\hat{p}, 1-\hat{p}]$*, they both choose own-biased learning*

ii) If only one player's prior is outside of $[\hat{p}, 1-\hat{p}]$*, they both choose the source biased in the same direction as such player's prior.*

Proposition 11 *When both players' priors are within* $[\hat{p}, 1-\hat{p}]$ *there are two Nash Equilibria:*

i) If both are biased in the same direction, they play a Stag Hunt Game

ii) If biased in opposite directions, they play a Battle of the Sexes.

These propositions show that if players are sufficiently central, they may pick a news source with a bias opposing their initial belief. This also shows that an agent may make different decisions, depending on who she expects to exchange news with in the next period. If agents play this sort of game with different players over time, one may see players choosing sources with different biases. Therefore, one can see this as one possible mechanism to explain multi homing behavior (switching back and forth between sources) of agents over time. As shown in Proposition 9, as sources' quality decreases, less priors find it optimal to coordinate their news choice. This means less people look at opposite biased news in equilibrium, if news quality in the market drops.

6.1.1 The role of sequential choice

So far we assumed that the choice of source was simultaneous. However, sometimes it may be more natural to consider sequential choices of news source (with or without the corresponding sequential observation of their message). This is, first, interesting as a robustness check of our predictions above; but it can also be interesting in itself to answer questions of the kind: what happens with learning outcomes when extreme agents are the first to choose information? what if the central agents are the first movers?

Not seeing the message: If both agents are central and one moves first in the news source choice (but both observe the message only at the end), the first mover understands that this affects the second mover's decision, who has a unique best-response. Thus, the first mover, internalizing this, has a unique optimal choice of source too (own-biased). This would refine the equilibrium, favoring the first mover.

Seeing the message: If both players are central, one moves first in the source choice and the message from the news source is commonly observed before the second player chooses, the first mover understands that this affects the second player's decision (who has a unique best-response to each realization, namely, own-biased learning from the interim belief). Then the first mover, internalizing this, has a more difficult problem: he needs to forecast the interim beliefs of the other player and her action at such interim belief, to base his choice of source on this forecast. In the baseline model (with disruptive signals), the equilibrium is unique in a similar way as above.

7 Discussion

When learning, agents usually have partial control over the information that they receive. This is in contrast to the majority of the literature studying optimal learning from biased sources, which either gives the agent no control or complete control over her information structure. In this paper, we include this natural feature and we highlight the importance of incorporating expectations about future information in order to understand agents' optimal learning choices. First, we show that for any belief that an agent holds, there exists some structure of the expected additional information that can change the source of information that she chooses. In addition, in the context of news bias, the accuracy and bias of the additional information has predictable effects on the news choice of different agents (i.e. with different beliefs). Apart from studying the impact of expecting exogenous information on learning choices, we also apply our framework to a situation where the expected additional information is endogenous and strategically chosen by other agents.

In the context of media consumption, considering this new perspective for news reading and sharing, allows us to identify a novel mechanism that may cause people to consume different types of biased news. Interestingly, agents would take into account *both* their bias and the bias of their peers for their learning decisions. Our framework suggest that, when people's priors are biased in different directions, both reading opposite and own-biased information can be rationalized for some combinations of priors.

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Appendix A.

Proof for Proposition 1:

Proof. For any p_0 we construct σ_e and σ_e' that makes σ^L and σ^R optimal respectively.
By the assumed condition, $\exists p$ such that $EU(\sigma^L|p) > EU(\sigma^R|p)$

Case 1:

If $p < p_0$ construct a binary signal, σ_e, with posteriors p and 1 ($\pi(l|L) = 1$, solve for $\pi(r|R)$).

$$\sum_{s\in\{l,r\}} \mathbb{P}(s|p_0)\Big(EU(\sigma^L|p(s^e)) - EU(\sigma^R|p(s^e))\Big) =$$
$$\mathbb{P}(l|p_0)\underbrace{\Big(EU(\sigma^L|p) - EU(\sigma^R|p)\Big)}_{>0} + \mathbb{P}(r|p_0)\underbrace{\Big(EU(\sigma^L|1) - EU(\sigma^R|1)\Big)}_{0} > 0$$

By observation 1, this implies $EU(\sigma^L, \sigma_e|p_0) > EU(\sigma^R, \sigma_e|p_0)$

Case 2:

If $p > p_0$ construct a binary signal, σ_e, with posteriors p and 0 ($\pi(r|R) = 1$, solve for $\pi(l|L)$).
By same logic as in case 1, $EU(\sigma^L, \sigma_e|p_0) > EU(\sigma^R, \sigma_e|p_0)$

Case 3:

If $p = p_0$ construct a binary signal that is just noise (σ^e will not influence beliefs and at current belief σ^L is optimal).
Construction for σ_e' follows same procedure.

■

Note: this proof is probably generalizable to the finite state, general action setting. Idea: construct external signal with one more signal realization than states. Signals realizations of external signal are perfectly informative about the state for all but one realization. For that realization, signal induces the belief for which the signal is optimal.

The following function of p will be useful for some of the following proofs. Note that it is continuous in p.

$$EU(\sigma^L|p) - EU(\sigma^R|p) =$$
$$= \begin{cases} 0 & p < \frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)} \\ \pi^L(r|R)p - \pi^L(r|L)(1-p) & \frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)} < p < \frac{\pi^R(r|L)}{\pi^R(r|L)+\pi^R(r|R)} \\ (\pi^L(l|L) - \pi^R(l|L))(1-p) - (\pi^R(r|R) - \pi^L(r|R))p & \frac{\pi^R(r|L)}{\pi^R(r|L)+\pi^R(r|R)} < p < \frac{\pi^L(l|L)}{\pi^L(l|L)+\pi^L(l|R)} \\ \pi^R(l|R)p - \pi^R(l|L)(1-p) & \frac{\pi^L(l|L)}{\pi^L(l|L)+\pi^L(l|R)} < p < \frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)} \\ 0 & p > \frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)} \end{cases}$$

Proof of Proposition 5.

Proof. First, we want to show that: $\frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R)}{\pi^R(r|L)} > \max\left\{\frac{\pi^e(r|R)}{\pi^e(r|L)}, \frac{\pi^e(l|L)}{\pi^e(l|R)}\right\} \implies$ "$p_0 = \frac{1}{2} \implies p(l^e), p(r^e) \in \left(\frac{\pi^R(r|L)}{\pi^R(r|L)+\pi^R(r|R)}, \frac{\pi^L(l|L)}{\pi^L(l|L)+\pi^L(l|R)}\right)$".

If $p_0 = \frac{1}{2}$, using the assumption that $\pi^X(l|L) + \pi^X(r|R) > 1$,

$$p(l^e) = \frac{\pi^e(l|R)}{\pi^e(l|R) + \pi^e(l|L)} < \frac{1}{2} < \frac{\pi^L(l|L)}{\pi^L(l|L) + \pi^L(l|R)}$$

$$p(r^e) = \frac{\pi^e(r|R)}{\pi^e(r|L) + \pi^e(r|R)} > \frac{1}{2} > \frac{\pi^R(r|L)}{\pi^R(r|L) + \pi^R(r|R)}$$

Then, it is left to show that:

$$p(l^e) > \frac{\pi^R(r|L)}{\pi^R(r|L) + \pi^R(r|R)} \iff \frac{\pi^R(r|R)}{\pi^R(r|L)} > \frac{\pi^e(l|L)}{\pi^e(l|R)}$$

$$p(r^e) < \frac{\pi^L(l|L)}{\pi^L(l|L) + \pi^L(l|R)} \iff \frac{\pi^e(r|R)}{\pi^e(r|L)} < \frac{\pi^L(l|L)}{\pi^L(l|R)}$$

which are implied by $\frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R)}{\pi^R(r|L)} > \frac{\pi^e(r|R)}{\pi^e(r|L)}, \frac{\pi^e(l|L)}{\pi^e(l|R)}$, completing the first part of the proof.

Next, we want to show that if $\frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R)}{\pi^R(r|L)} > \max\left\{\frac{\pi^e(r|R)}{\pi^e(r|L)}, \frac{\pi^e(l|L)}{\pi^e(l|R)}\right\}$, there exists $\epsilon > 0$ s.t. $\forall p \in N_\epsilon\left(\frac{1}{2}\right)$, if the DM's prior is $p_0 = p$, own-biased learning is optimal.

Since $p(l^e)$ and $p(r^e)$ are continuous in p_0, what we showed above for $\frac{1}{2}$ is also true for some priors in the neighborhood of $\frac{1}{2}$. More formally, $\exists \epsilon > 0$ s.t. if $p \in N_\epsilon\left(\frac{1}{2}\right)$, and $p_0 = p$, then $p(l^e), p(r^e) \in \left[\frac{\pi^R(r|L)}{\pi^R(r|L)+\pi^R(r|R)}, \frac{\pi^L(l|L)}{\pi^L(l|L)+\pi^L(l|R)}\right]$.

Then, using our observation 1,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = \mathbb{P}(l^e|p)(EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e)))+$$

$$\mathbb{P}(r^e|p)(EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e)))$$

and plugging in for $EU(\sigma^L|p) - EU(\sigma^R|p)$, given where the interim posteriors lie,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = \mathbb{P}(l^e|p)((\pi^L(l|L) - \pi^R(l|L))(1 - p(l^e)) - (\pi^R(r|R) - \pi^L(r|R))p(l^e))+$$

$$\mathbb{P}(r^e|p)((\pi^L(l|L) - \pi^R(l|L))(1 - p(r^e)) - (\pi^R(r|R) - \pi^L(r|R))p(r^e)).$$

Finally, using $\mathbb{P}(l^e|p) = \pi^e(l|R)p + \pi^e(l|L)(1-p)$ and $\mathbb{P}(r^e|p) = \pi^e(r|R)p + \pi^e(r|L)(1-p)$, as well as $\pi^L(l|L) - \pi^R(l|L) = \pi^R(r|R) - \pi^L(r|R)$ by symmetry, and plugging in for the interim posteriors $p(l^e) = \frac{\pi^e(l|R)p_0}{\pi^e(l|R)p_0+\pi^e(l|L)(1-p_0)}$ and $p(r^e) = \frac{\pi^e(r|R)p_0}{\pi^e(r|R)p_0+\pi^e(r|L)(1-p_0)}$, one obtains

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = (\pi^L(l|L) - \pi^R(l|L))(1 - 2p)$$

which is positive i.f.f. $p < \frac{1}{2}$, negative i.f.f. $p > \frac{1}{2}$ and equal to zero i.f.f. $p = \frac{1}{2}$. This completes the proof. ■

The following proof is for Lemma 1.

Proof. For the first part of the statement,

$$\frac{\pi^x(r|R)}{\pi^x(r|L)} > \frac{\pi^x(l|L)}{\pi^x(l|R)} \iff \pi^x(r|R)(1 - \pi^x(r|R)) > \pi^x(l|L)(1 - \pi^x(l|L))$$

Since $x(1-x)$ is an inverted symmetric parabola maximized at $x = \frac{1}{2}$, the statement above will be true as long as $d\left(\pi^x(r|R), \frac{1}{2}\right) < d\left(\pi^x(l|L), \frac{1}{2}\right)$. Then, our assumption that $\pi^x(r|R) + \pi^x(l|L) > 1$ implies that $d\left(\pi^x(r|R), \frac{1}{2}\right) < d\left(\pi^x(l|L), \frac{1}{2}\right) \iff \pi^x(r|R) < \pi^x(l|L)$, which is equivalent to σ^x being left-biased. Similarly,

$$\frac{\pi^x(l|L)}{\pi^x(l|R)} > \frac{\pi^x(r|R)}{\pi^x(r|L)} \iff \pi^x(l|L)(1 - \pi^x(l|L)) > \pi^x(r|R)(1 - \pi^x(r|R))$$

Since $x(1-x)$ is an inverted symmetric parabola maximized at $x = \frac{1}{2}$, the statement above will be true as long as $d\left(\pi^x(l|L), \frac{1}{2}\right) < d\left(\pi^x(r|R), \frac{1}{2}\right)$. Then, our assumption that $\pi^x(r|R) + \pi^x(l|L) > 1$ implies that $d\left(\pi^x(l|L), \frac{1}{2}\right) < d\left(\pi^x(r|R), \frac{1}{2}\right) \iff \pi^x(l|L) < \pi^x(r|R)$, which is equivalent to σ^x being right-biased.

Finally,

$$\frac{\pi^x(r|R)}{\pi^x(r|L)} = \frac{\pi^x(l|L)}{\pi^x(l|R)} \iff \pi^x(r|R)(1 - \pi^x(r|R)) = \pi^x(l|L)(1 - \pi^x(l|L))$$

which by $\pi^x(r|R) + \pi^x(l|L) > 1$, requires $\pi^x(r|R) = \pi^x(l|L) > \frac{1}{2}$ that implies σ^x being unbiased. ■

Lemma 4 *If* $\frac{\pi^e(l|L)}{\pi^e(l|R)} > \frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)}$, *then,* σ_e *is right-biased and there exists an interval* $[\underline{p}, \bar{p}]$ *such that* $\frac{1}{2} \in [\underline{p}, \bar{p}]$ *and, if* $p_0 \in [\underline{p}, \bar{p}]$ *the DM finds it optimal to choose* σ^R, *that is,* $EU(\sigma^L, \sigma_e|p_0) < EU(\sigma^R, \sigma_e|p_0)$.

Proof. For the first part of the statement, note that

$$\frac{\pi^e(l|L)}{\pi^e(l|R)} > \frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)} \implies \frac{\pi^e(l|L)}{\pi^e(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)}$$

which by Lemma 1 implies that σ_e is right-biased.

For the second part, note that if $p_0 = \frac{1}{2}$,

$$p(l^e) < \frac{\pi^L(r|L)}{\pi^L(r|L) + \pi^L(r|R)} \iff \frac{\pi^e(l|L)}{\pi^e(l|R)} > \frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)}$$

and

$$p(r^e) < \frac{\pi^R(l|L)}{\pi^R(l|L) + \pi^R(l|R)} \iff \frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)}.$$

Then, we want to show that if $\frac{\pi^e(l|L)}{\pi^e(l|R)} > \frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)}$, there exists $\epsilon > 0$ s.t. $\forall p \in N_\epsilon\left(\frac{1}{2}\right)$, if the DM's prior is $p_0 = p$, $EU(\sigma^L, \sigma_e|p_0) < EU(\sigma^R, \sigma_e|p_0)$.

Since $p(l^e)$ and $p(r^e)$ are continuous in p_0, what we showed above for $\frac{1}{2}$ is also true for some priors in the neighborhood of $\frac{1}{2}$. More formally, $\exists\epsilon > 0$ s.t. if $p \in N_\epsilon\left(\frac{1}{2}\right)$, and $p_0 = p$, then $p(l^e) < \frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}$ and $\frac{1}{2} < p(r^e) < \frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}$.

Then, using our observation 1,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = \mathbb{P}(l^e|p)(EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e)))+$$

$$\mathbb{P}(r^e|p)(EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e)))$$

and plugging in for $EU(\sigma^L|p) - EU(\sigma^R|p)$ given where the interim posteriors lie, as well as $\pi^L(l|L) - \pi^R(l|L) = \pi^R(r|R) - \pi^L(r|R)$ by symmetry,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = \mathbb{P}(r^e|p)(EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e))) =$$

$$= \begin{cases} \mathbb{P}(r^e|p)(\pi^L(l|L) - \pi^L(r|R))(1 - 2p(r^e)) & \frac{1}{2} < p(r^e) < \frac{\pi^L(l|L)}{\pi^L(l|L)+\pi^L(l|R)} \\ \mathbb{P}(r^e|p)(\pi^R(l|R)p(r^e) - \pi^R(l|L)(1 - p(r^e))) & \frac{\pi^L(l|L)}{\pi^L(l|L)+\pi^L(l|R)} < p(r^e) < \frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)} \end{cases}$$

By σ^L being left-biased and $p(r^e) > \frac{1}{2}$, $\mathbb{P}(r^e|p)(\pi^L(l|L) - \pi^L(r|R))(1 - 2p(r^e)) < 0$. Moreover, $\mathbb{P}(r^e|p)(\pi^R(l|R)p(r^e) - \pi^R(l|L)(1 - p(r^e))) < 0 \iff \frac{\pi^R(r|R)p}{\pi^e(r|L)(1-p)} < \frac{\pi^R(l|L))}{\pi^R(l|R)}$, which by $\frac{\pi^e(l|L)}{\pi^e(l|R)} > \frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)}$ is true for $p = \frac{1}{2}$ and, by continuity, it is also true for p in some neighborhood of $\frac{1}{2}$.

Therefore, we can conclude that there exists $\epsilon > 0$ s.t. $\forall p \in N_\epsilon\left(\frac{1}{2}\right)$, if the DM's prior is $p_0 = p$, $EU(\sigma^L, \sigma_e|p_0) < EU(\sigma^R, \sigma_e|p_0)$. ■

Lemma 5 *If* $\frac{\pi^e(r|R)}{\pi^e(r|L)} > \frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(l|L)}{\pi^e(l|R)}$*, then,* σ_e *is left-biased and there exists an interval* $[\underline{p}, \bar{p}]$ *such that* $\frac{1}{2} \in [\underline{p}, \bar{p}]$ *and, if* $p_0 \in [\underline{p}, \bar{p}]$ *the DM finds it optimal to choose* σ^L*, that is,* $EU(\sigma^L, \sigma_e|p_0) > EU(\sigma^R, \sigma_e|p_0)$*.*

Proof. For the first part of the statement, note that

$$\frac{\pi^e(r|R)}{\pi^e(r|L)} > \frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(l|L)}{\pi^e(l|R)} \implies \frac{\pi^e(r|R)}{\pi^e(r|L)} > \frac{\pi^e(l|L)}{\pi^e(l|R)}$$

which by Lemma 1 implies that σ_e is left-biased.

For the second part, note that if $p_0 = \frac{1}{2}$,

$$p(l^e) > \frac{\pi^L(r|L)}{\pi^L(r|L) + \pi^L(r|R)} \iff \frac{\pi^e(l|L)}{\pi^e(l|R)} < \frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)}$$

and

$$p(r^e) > \frac{\pi^R(l|L)}{\pi^R(l|L) + \pi^R(l|R)} \iff \frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} < \frac{\pi^e(r|R)}{\pi^e(r|L)}.$$

Then, we want to show that if $\frac{\pi^e(r|R)}{\pi^e(r|L)} > \frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(l|L)}{\pi^e(l|R)}$, there exists $\epsilon > 0$ s.t. $\forall p \in N_\epsilon\left(\frac{1}{2}\right)$, if the DM's prior is $p_0 = p$, $EU(\sigma^L, \sigma_e|p_0) > EU(\sigma^R, \sigma_e|p_0)$.

Since $p(l^e)$ and $p(r^e)$ are continuous in p_0, what we showed above for $\frac{1}{2}$ is also true for some priors in the neighborhood of $\frac{1}{2}$. More formally, $\exists \epsilon > 0$ s.t. if $p \in N_\epsilon\left(\frac{1}{2}\right)$, and $p_0 = p$, then $\frac{1}{2} > p(l^e) > \frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}$ and $p(r^e) > \frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}$.

Then, using our observation 1,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = \mathbb{P}(l^e|p)(EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e))) +$$

$$\mathbb{P}(r^e|p)(EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e)))$$

and plugging in for $EU(\sigma^L|p) - EU(\sigma^R|p)$ given where the interim posteriors lie, as well as $\pi^L(l|L) - \pi^R(l|L) = \pi^R(r|R) - \pi^L(r|R)$ by symmetry,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = \mathbb{P}(l^e|p)(EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e))) =$$

$$= \begin{cases} \mathbb{P}(l^e|p)(\pi^L(r|R)p(l^e) - \pi^L(r|L)(1 - p(l^e))) & \frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)} < p(l^e) < \frac{\pi^R(r|L)}{\pi^R(r|L)+\pi^R(r|R)} \\ \mathbb{P}(l^e|p)(\pi^L(l|L) - \pi^L(r|R))(1 - 2p(l^e)) & \frac{\pi^R(r|L)}{\pi^R(r|L)+\pi^R(r|R)} < p(l^e) < \frac{1}{2} \end{cases}$$

By σ^L being left-biased and $p(l^e) < \frac{1}{2}$, $\mathbb{P}(l^e|p)(\pi^L(l|L) - \pi^L(r|R))(1 - 2p(l^e)) > 0$. Moreover, $\mathbb{P}(l^e|p)(\pi^L(r|R)p(l^e) - \pi^L(r|L)(1 - p(l^e))) > 0 \iff \frac{\pi^e(l|L)(1-p)}{\pi^e(l|R)p} < \frac{\pi^L(r|R)}{\pi^L(r|L)}$ is true for $p = \frac{1}{2}$ and, by continuity, it is also true for p in some neighborhood of $\frac{1}{2}$.

Therefore, we can conclude that there exists $\epsilon > 0$ s.t. $\forall p \in N_\epsilon\left(\frac{1}{2}\right)$, if the DM's prior is $p_0 = p$, $EU(\sigma^L, \sigma_e|p_0) > EU(\sigma^R, \sigma_e|p_0)$. ■

Lemma 6 *If $\frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(l|L)}{\pi^e(l|R)} > \frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R))}{\pi^R(r|L)}$ and $\frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)} > \frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R))}{\pi^R(r|L)}$, then, there exists an interval $[\underline{p}, \bar{p}]$ such that $\frac{1}{2} \in [\underline{p}, \bar{p}]$ and, if $p_0 \in [\underline{p}, \bar{p}]$ the DM finds it optimal to choose the source that is biased in the same direction as the exogenous source, that is,*

$$EU(\sigma^L, \sigma_e|p_0) > EU(\sigma^R, \sigma_e|p_0) \iff \pi^e(l|L) > \pi^e(r|R) \text{ and}$$

$$EU(\sigma^L, \sigma_e|p_0) < EU(\sigma^R, \sigma_e|p_0) \iff \pi^e(l|L) < \pi^e(r|R).$$

Proof.

First, we will argue that $\frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(l|L)}{\pi^e(l|R)} > \frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R))}{\pi^R(r|L)}$ and $\frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)} > \frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R))}{\pi^R(r|L)}$ implies that if $p_0 = \frac{1}{2}$, then

$$p(l^e) \in \left[\frac{\pi^L(r|L)}{\pi^L(r|L) + \pi^L(r|R)}, \frac{\pi^R(r|L)}{\pi^R(r|L) + \pi^R(r|R)}\right]$$

$$p(r^e) \in \left[\frac{\pi^L(l|L)}{\pi^L(l|L) + \pi^L(l|R)}, \frac{\pi^R(l|L)}{\pi^R(l|L) + \pi^R(l|R)}\right]$$

To see this, note that:

$$p(l^e) > \frac{\pi^L(r|L)}{\pi^L(r|L) + \pi^L(r|R)} \iff \frac{\pi^L(r|R)}{\pi^L(r|L)} > \frac{\pi^e(l|L)}{\pi^e(l|R)},$$

$$p(l^e) < \frac{\pi^R(r|L)}{\pi^R(r|L) + \pi^R(r|R)} \iff frac\pi^e(l|L)\pi^e(l|R) > \frac{\pi^R(r|R))}{\pi^R(r|L)},$$

$$p(r^e) > \frac{\pi^L(l|L)}{\pi^L(l|L) + \pi^L(l|R)} \iff \frac{\pi^e(r|R)}{\pi^e(r|L)} > \frac{\pi^L(l|L)}{\pi^L(l|R)}$$

and

$$p(r^e) < \frac{\pi^R(l|L)}{\pi^R(l|L) + \pi^R(l|R)} \iff \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)}.$$

Then, we want to show that if $\frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(l|L)}{\pi^e(l|R)} > \frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R))}{\pi^R(r|L)}$ and $\frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)} > \frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R))}{\pi^R(r|L)}$, there exists $\epsilon > 0$ s.t. $\forall p \in N_\epsilon\left(\frac{1}{2}\right)$, if the DM's prior is $p_0 = p$, $EU(\sigma^L, \sigma_e|p_0) > EU(\sigma^R, \sigma_e|p_0) \iff \sigma_e$ is left-biased, that is, $\pi^e(l|L) > \pi^e(r|R)$ and $EU(\sigma^R, \sigma_e|p_0) > EU(\sigma^L, \sigma_e|p_0) \iff \sigma_e$ is right-biased, that is, $\pi^e(r|R) > \pi^e(l|L)$.

Since $p(l^e)$ and $p(r^e)$ are continuous in p_0, $\exists \epsilon > 0$ s.t. if $p \in N_\epsilon\left(\frac{1}{2}\right)$, and $p_0 = p$, then

$$p(l^e) \in \left[\frac{\pi^L(r|L)}{\pi^L(r|L) + \pi^L(r|R)}, \frac{\pi^R(r|L)}{\pi^R(r|L) + \pi^R(r|R)}\right]$$

$$p(r^e) \in \left[\frac{\pi^L(l|L)}{\pi^L(l|L) + \pi^L(l|R)}, \frac{\pi^R(l|L)}{\pi^R(l|L) + \pi^R(l|R)}\right].$$

As before, using observation 1,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = \mathbb{P}(l^e|p)(EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e)))+$$

$$\mathbb{P}(r^e|p)(EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e)))$$

and plugging in for $EU(\sigma^L|p) - EU(\sigma^R|p)$ given where the interim posteriors lie,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = \mathbb{P}(l^e|p)(\pi^L(r|R)p(l^e) - \pi^L(r|L)(1 - p(l^e)))+$$

$$\mathbb{P}(r^e|p)(\pi^R(l|R)p(r^e) - \pi^R(l|L)(1 - p(r^e))) =$$

$$= \pi^L(r|R)(\pi^e(l|R)p - \pi^e(r|L)(1-p)) + \pi^L(r|L)(\pi^e(r|R)p - \pi^e(l|L)(1-p)) =$$

$$= (2p-1)\pi^L(r|R) + (\pi^e(l|L)(1-p) - \pi^e(r|R)p)(\pi^L(r|R) + \pi^L(l|L) - 1)$$

Then, note that for $p = \frac{1}{2}$, this expression is positive iff σ_e is left-biased. Namely,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = \frac{1}{2}(\pi^e(l|L) - \pi^e(r|R))(\pi^L(r|R) + \pi^L(l|L) - 1) > 0 \iff \pi^e(l|L) > \pi^e(r|R)$$

By continuity, this is also true for p in some neighborhood of $\frac{1}{2}$. Therefore, we can conclude that there exists $\epsilon > 0$ s.t. $\forall p \in N_\epsilon\left(\frac{1}{2}\right)$, if the DM's prior is $p_0 = p$, $EU(\sigma^L, \sigma_e|p_0) > EU(\sigma^R, \sigma_e|p_0) \iff \pi^e(l|L) > \pi^e(r|R)$. Using the same logic, there exists $\epsilon > 0$ s.t. $\forall p \in N_\epsilon\left(\frac{1}{2}\right)$, if the DM's prior is $p_0 = p$, $EU(\sigma^L, \sigma_e|p_0) < EU(\sigma^R, \sigma_e|p_0) \iff \pi^e(l|L) < \pi^e(r|R)$.
■

Lemma 7 *If* $\frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(l|L)}{\pi^e(l|R)} > \frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R))}{\pi^R(r|L)} > \frac{\pi^e(r|R)}{\pi^e(r|L)}$, *then,* σ_e *is right-biased and there exists an interval* $[\underline{p}, \bar{p}]$ *such that* $\frac{1}{2} \in [\underline{p}, \bar{p}]$ *and, if* $p_0 \in [\underline{p}, \bar{p}]$ *the DM finds it weakly optimal to choose* σ^R, *that is,* $EU(\sigma^L, \sigma_e|p_0) < EU(\sigma^R, \sigma_e|p_0)$.

Proof. For the first part of the statement, note that

$$\frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(l|L)}{\pi^e(l|R)} > \frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R))}{\pi^R(r|L)} > \frac{\pi^e(r|R)}{\pi^e(r|L)} \implies \frac{\pi^e(l|L)}{\pi^e(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)}$$

which by Lemma 1 implies that σ_e is right-biased.

For the second part, note that if $p_0 = \frac{1}{2}$,

$$p(l^e) < \frac{\pi^R(r|L)}{\pi^R(r|L) + \pi^R(r|R)} \iff \frac{\pi^e(l|L)}{\pi^e(l|R)} > \frac{\pi^R(r|R)}{\pi^L R(r|L)},$$

$$p(l^e) > \frac{\pi^L(r|L)}{\pi^L(r|L) + \pi^L(r|R)} \iff \frac{\pi^L(r|R))}{\pi^L(r|L)} > \frac{\pi^e(l|L)}{\pi^e(l|R)},$$

$$p(r^e) > \frac{1}{2}$$

and

$$p(r^e) < \frac{\pi^L(l|L)}{\pi^L(l|L) + \pi^L(l|R)} \iff \frac{\pi^L(l|L)}{\pi^L(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)}.$$

This means that when$p_0 = \frac{1}{2}$, $\frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(l|L)}{\pi^e(l|R)} > \frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R))}{\pi^R(r|L)} > \frac{\pi^e(r|R)}{\pi^e(r|L)} \implies$

$$p(l^e) \in \left[\frac{\pi^L(r|L)}{\pi^L(r|L) + \pi^L(r|R)}, \frac{\pi^R(r|L)}{\pi^R(r|L) + \pi^R(r|R)}\right]$$

$$p(r^e) \in \left[\frac{1}{2}, \frac{\pi^L(l|L)}{\pi^L(l|L) + \pi^L(l|R)}\right]$$

Then, we want to show that if $\frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(l|L)}{\pi^e(l|R)} > \frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R))}{\pi^R(r|L)} > \frac{\pi^e(r|R)}{\pi^e(r|L)}$, there exists $\epsilon > 0$ s.t. $\forall p \in N_\epsilon\left(\frac{1}{2}\right)$, if the DM's prior is $p_0 = p$, $EU(\sigma^R, \sigma_e|p_0) > EU(\sigma^L, \sigma_e|p_0)$.

Since $p(l^e)$ and $p(r^e)$ are continuous in p_0, what we showed above for $\frac{1}{2}$ is also true for some priors in the neighborhood of $\frac{1}{2}$. More formally, $\exists\epsilon > 0$ s.t. if $p \in N_\epsilon\left(\frac{1}{2}\right)$, and $p_0 = p$, then

$$p(l^e) \in \left[\frac{\pi^L(r|L)}{\pi^L(r|L) + \pi^L(r|R)}, \frac{\pi^R(r|L)}{\pi^R(r|L) + \pi^R(r|R)}\right]$$

and

$$p(r^e) \in \left[\frac{1}{2}, \frac{\pi^L(l|L)}{\pi^L(l|L) + \pi^L(l|R)}\right].$$

Then, using our observation 1,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = \mathbb{P}(l^e|p)(EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e)))+$$

$$\mathbb{P}(r^e|p)(EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e)))$$

and plugging in for $EU(\sigma^L|p) - EU(\sigma^R|p)$ given where the interim posteriors lie, as well as $\pi^L(l|L) - \pi^R(l|L) = \pi^R(r|R) - \pi^L(r|R)$ by symmetry,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = \mathbb{P}(l^e|p)(\pi^L(r|R)p(l^e) - \pi^L(r|L)(1 - p(l^e)))+$$

$$\mathbb{P}(r^e|p)(\pi^L(l|L) - \pi^L(r|R))(1 - 2p(r^e)) =$$

$$= p(\pi^L(r|R) - \pi^L(l|L)\pi^e(r|R)) + (1 - p)(\pi^L(l|L) - \pi^e(l|L) - \pi^L(r|R)\pi^e(r|L))$$

Then, note that for $p = \frac{1}{2}$, this expression is strictly negative and therefore $EU(\sigma^R, \sigma^e|p) > EU(\sigma^L, \sigma^e|p)$. Namely, if $p = \frac{1}{2}$,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = \frac{1}{2}(\pi^L(l|L)\pi^e(l|R) - \pi^L(l|R)\pi^e(l|L)) \geq 0 \iff \frac{\pi^L(l|L)}{\pi^L(l|R)} \geq \frac{\pi^e(l|L)}{\pi^e(l|R)}$$

which is a contradiction. By continuity, it is also true for p in some neighborhood of $\frac{1}{2}$.

Therefore, we can conclude that there exists $\epsilon > 0$ s.t. $\forall p \in N_\epsilon\left(\frac{1}{2}\right)$, if the DM's prior is $p_0 = p$, $EU(\sigma^R, \sigma_e|p_0) > EU(\sigma^L, \sigma_e|p_0)$. ■

Lemma 8 *If* $\frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)} > \frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R))}{\pi^R(r|L)} > \frac{\pi^e(l|L)}{\pi^e(l|R)}$, *then,* σ_e *is left-biased and there exists an interval* $[\underline{p}, \bar{p}]$ *such that* $\frac{1}{2} \in [\underline{p}, \bar{p}]$ *and, if* $p_0 \in [\underline{p}, \bar{p}]$ *the DM finds it weakly optimal to choose* σ^L*, that is,* $EU(\sigma^L, \sigma_e|p_0) > EU(\sigma^R, \sigma_e|p_0)$.

Proof. For the first part of the statement, note that

$$\frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)} > \frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R))}{\pi^R(r|L)} > \frac{\pi^e(l|L)}{\pi^e(l|R)} \implies \frac{\pi^e(r|R)}{\pi^e(r|L)} > \frac{\pi^e(l|L)}{\pi^e(l|R)}$$

which by Lemma 1 implies that σ_e is left-biased.

For the second part, note that if $p_0 = \frac{1}{2}$,

$$p(l^e) < \frac{1}{2},$$

$$p(l^e) > \frac{\pi^R(r|L)}{\pi^R(r|L) + \pi^R(r|R)} \iff \frac{\pi^R(r|R))}{\pi^R(r|L)} > \frac{\pi^e(l|L)}{\pi^e(l|R)},$$

$$p(r^e) > \frac{\pi^L(l|L)}{\pi^L(l|L) + \pi^L(l|R)} \iff \frac{\pi^e(r|R)}{\pi^e(r|L)} > \frac{\pi^L(l|L)}{\pi^L(l|R)}$$

and

$$p(r^e) < \frac{\pi^R(l|L)}{\pi^R(l|L) + \pi^R(l|R)} \iff \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)}.$$

This means that when$p_0 = \frac{1}{2}$, $\frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)} > \frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R))}{\pi^R(r|L)} > \frac{\pi^e(l|L)}{\pi^e(l|R)} \implies$

$$p(l^e) \in \left[\frac{\pi^R(r|L)}{\pi^R(r|L) + \pi^R(r|R)}, \frac{1}{2}\right]$$

$$p(r^e) \in \left[\frac{\pi^L(l|L)}{\pi^L(l|L) + \pi^L(l|R)}, \frac{\pi^R(l|L)}{\pi^R(l|L) + \pi^R(l|R)}\right]$$

Then, we want to show that if $\frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} > \frac{\pi^e(r|R)}{\pi^e(r|L)} > \frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R))}{\pi^R(r|L)} > \frac{\pi^e(l|L)}{\pi^e(l|R)}$, there exists $\epsilon > 0$ s.t. $\forall p \in N_\epsilon\left(\frac{1}{2}\right)$, if the DM's prior is $p_0 = p$, $EU(\sigma^L, \sigma_e|p_0) > EU(\sigma^R, \sigma_e|p_0)$.

Since $p(l^e)$ and $p(r^e)$ are continuous in p_0, what we showed above for $\frac{1}{2}$ is also true for some priors in the neighborhood of $\frac{1}{2}$. More formally, $\exists \epsilon > 0$ s.t. if $p \in N_\epsilon\left(\frac{1}{2}\right)$, and $p_0 = p$, then

$$p(l^e) \in \left[\frac{\pi^R(r|L)}{\pi^R(r|L) + \pi^R(r|R)}, \frac{1}{2}\right]$$

and

$$p(r^e) \in \left[\frac{\pi^L(l|L)}{\pi^L(l|L) + \pi^L(l|R)}, \frac{\pi^R(l|L)}{\pi^R(l|L) + \pi^R(l|R)}\right].$$

Then, using our observation 1,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = \mathbb{P}(l^e|p)(EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e)))+$$

$$\mathbb{P}(r^e|p)(EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e)))$$

and plugging in for $EU(\sigma^L|p) - EU(\sigma^R|p)$ given where the interim posteriors lie, as well as $\pi^L(l|L) - \pi^R(l|L) = \pi^R(r|R) - \pi^L(r|R)$ by symmetry,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = \mathbb{P}(l^e|p)(\pi^L(l|L) - \pi^L(r|R))(1 - 2p(l^e))+$$

$$\mathbb{P}(r^e|p)(\pi^L(r|L)p(r^e) - \pi^L(r|R)(1 - p(r^e)))$$

Then, note that for $p = \frac{1}{2}$, this expression is strictly positive and therefore $EU(\sigma^L, \sigma^e|p) > EU(\sigma^R, \sigma^e|p)$. Namely, if $p = \frac{1}{2}$,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = \frac{1}{2}(\pi^e(r|R)\pi^L(l|R) - \pi^L(l|L)\pi^e(r|L)) \leq 0 \iff \frac{\pi^e(r|R)}{\pi^e(r|L)} \leq \frac{\pi^L(l|L)}{\pi^L(l|R)}$$

which is a contradiction. By continuity, it is also true for p in some neighborhood of $\frac{1}{2}$.

Therefore, we can conclude that there exists $\epsilon > 0$ s.t. $\forall p \in N_\epsilon\left(\frac{1}{2}\right)$, if the DM's prior is

$p_0 = p$, $EU(\sigma^L, \sigma_e|p_0) > EU(\sigma^R, \sigma_e|p_0)$. ■

Since the above lemmas cover all cases in which $\max\left\{\frac{\pi^e(r|R)}{\pi^e(r|L)}, \frac{\pi^e(l|L)}{\pi^e(l|R)}\right\} > \frac{\pi^L(l|L)}{\pi^L(l|R)} = \frac{\pi^R(r|R)}{\pi^R(r|L)}$ and $\frac{\pi^R(l|L)}{\pi^R(l|R)} = \frac{\pi^L(r|R)}{\pi^L(r|L)} > \min\left\{\frac{\pi^e(r|R)}{\pi^e(r|L)}, \frac{\pi^e(l|L)}{\pi^e(l|R)}\right\}$,they are enough to prove 6.

Proof of Proposition 4.

Proof. First, we want to show that: $\frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} < \min\left\{\frac{\pi^e(r|R)}{\pi^e(r|L)}, \frac{\pi^e(l|L)}{\pi^e(l|R)}\right\} \implies$ "$p_0 = \frac{1}{2} \implies p(l^e) < \frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}, p(r^e) > \frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}$".

If $p_0 = \frac{1}{2}$,

$$p(l^e) = \frac{\pi^e(l|R)}{\pi^e(l|R) + \pi^e(l|L)} < \frac{\pi^L(r|L)}{\pi^L(r|L) + \pi^L(r|R)} \iff \frac{\pi^L(r|R)}{\pi^L(r|L)} < \frac{\pi^e(l|L)}{\pi^e(l|R)};$$

$$p(r^e) = \frac{\pi^e(r|R)}{\pi^e(r|L) + \pi^e(r|R)} > \frac{\pi^R(l|L)}{\pi^R(l|L) + \pi^R(l|R)} \iff \frac{\pi^R(l|L)}{\pi^R(l|R)} < \frac{\pi^e(r|R)}{\pi^e(r|L)}.$$

This completes the first part of the proof.

Next, we want to show that if $\frac{\pi^L(r|R)}{\pi^L(r|L)} = \frac{\pi^R(l|L)}{\pi^R(l|R)} < \min\left\{\frac{\pi^e(r|R)}{\pi^e(r|L)}, \frac{\pi^e(l|L)}{\pi^e(l|R)}\right\}$, there exists $\epsilon > 0$ s.t. $\forall p \in N_\epsilon\left(\frac{1}{2}\right)$, if the DM's prior is $p_0 = p$, the DM is indifferent between choosing σ^L and σ^R.

Since $p(l^e)$ and $p(r^e)$ are continuous in p_0, what we showed above for $\frac{1}{2}$ is also true for some priors in the neighborhood of $\frac{1}{2}$. More formally, $\exists \epsilon > 0$ s.t. if $p \in N_\epsilon\left(\frac{1}{2}\right)$, and $p_0 = p$, then $p(l^e) < \frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}, p(r^e) > \frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}$.

Then, using our observation 1,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = \mathbb{P}(l^e|p)(EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e)))+$$

$$\mathbb{P}(r^e|p)(EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e)))$$

and plugging in for $EU(\sigma^L|p) - EU(\sigma^R|p)$, given where the interim posteriors lie,

$$EU(\sigma^L, \sigma^e|p) - EU(\sigma^R, \sigma^e|p) = 0.$$

This completes the proof. ■

Proposition 3 is proved below.

Proof. Note that a DM whose interim posteriors are all either below $\frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}$ or above

$\frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}$ would be indifferent between the two sources. To see this, by observation 1,

$$EU(\sigma^L, \sigma^e|p_0) - EU(\sigma^R, \sigma^e|p_0) = \mathbb{P}(l^e|p_0)(EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e)))+$$

$$\mathbb{P}(r^e|p_0)(EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e)))$$

and by where the posteriors lie,

$$EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e)) = EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e)) = 0 \implies$$

$$EU(\sigma^L, \sigma^e|p_0) - EU(\sigma^R, \sigma^e|p_0) = 0.$$

Since by Bayes Rule $p(l^e) < p_0 < p(r^e)$ (as long as the exogenous source is informative), then for all the posteriors to be below $\frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}$ it is sufficient that

$$p(r^e) < \frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)} \iff p_0 < \frac{\pi^L(r|L)\pi^e(r|L)}{\pi^L(r|L)\pi^e(r|L)+\pi^L(r|R)\pi^e(r|R)}$$

and for all the posteriors to be above $\frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}$ it is sufficient that

$$p(l^e) > \frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)} \iff p_0 > \frac{\pi^L(l|L)\pi^e(l|L)}{\pi^L(l|L)\pi^e(l|L)+\pi^L(l|R)\pi^e(l|R)}$$

Then, letting $p_1 = \frac{\pi^L(r|L)\pi^e(r|L)}{\pi^L(r|L)\pi^e(r|L)+\pi^L(r|R)\pi^e(r|R)}$ and $p_6 = \frac{\pi^L(l|L)\pi^e(l|L)}{\pi^L(l|L)\pi^e(l|L)+\pi^L(l|R)\pi^e(l|R)}$ we complete the proof for case i). Note that $p_1 > 0 \iff \pi^L(r|L)\pi^e(r|L) > 0$ and $p_6 < 1 \iff \pi^L(l|R)\pi^e(l|R) > 0$. Also note that p_1 decreases with the informativeness of σ^e in the sense that $\frac{\partial p_1}{\partial \pi^e(l|L)} < 0$ and $\frac{\partial p_1}{\partial \pi^e(r|R)} < 0$. On the other hand, p_6 increases with the informativeness of σ^e in the sense that $\frac{\partial p_6}{\partial \pi^e(l|L)} > 0$ and $\frac{\partial p_6}{\partial \pi^e(r|R)} > 0$. This implies that the indifference regions become smaller the more informative the additional source is.

Next, a DM whose interim posteriors are all either below $\frac{1}{2}$ and at least one of them is above $\frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}$, or, above $\frac{1}{2}$ and at least one of them below $\frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}$, would find own-biased learning optimal.

To see this, by observation 1,

$$EU(\sigma^L, \sigma^e|p_0) - EU(\sigma^R, \sigma^e|p_0) = \mathbb{P}(l^e|p_0)(EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e)))+$$

$$\mathbb{P}(r^e|p_0)(EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e)))$$

and if all interim posteriors are below $\frac{1}{2}$ and at least one of them is above $\frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}$

$$EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e)) \geq 0, \;\; EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e)) > 0 \implies$$

$$EU(\sigma^L, \sigma^e|p_0) - EU(\sigma^R, \sigma^e|p_0) > 0.$$

while if all interim posteriors are above $\frac{1}{2}$ and at least one of them below $\frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}$

$$EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e)) < 0, \;\; EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e)) \leq 0 \implies$$

$$EU(\sigma^L, \sigma^e|p_0) - EU(\sigma^R, \sigma^e|p_0) < 0.$$

Again, using that $p(l^e) < p_0 < p(r^e)$, for all interim posteriors are below $\frac{1}{2}$ and at least one of them is above $\frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}$ it is enough that

$$\frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)} < p(r^e) < \frac{1}{2} \iff p_1 < p_0 < \frac{\pi^e(r|L)}{\pi^e(r|L)+\pi^e(r|R)}$$

and for all interim posteriors are above $\frac{1}{2}$ and at least one of them below $\frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}$ it is enough that

$$\frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)} > p(l^e) > \frac{1}{2} \iff p_5 > p_0 > \frac{\pi^e(l|L)}{\pi^e(l|L)+\pi^e(l|R)}.$$

Letting $p_2 = \frac{\pi^e(r|L)}{\pi^e(r|L)+\pi^e(r|R)}$ and $p_5 = \frac{\pi^e(l|L)}{\pi^e(l|L)+\pi^e(l|R)}$ we complete the proof of case ii). Note that $p_2 > p_1$ and $p_6 > p_5 \iff \pi^L(l|L) + \pi^L(r|R) > 1$ which is exactly assumption **??**.

For case iii), a DM whose left interim posterior is below $\frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}$ and the right interim posterior is between $\frac{1}{2}$ and $\frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}$, or, whose left interim posterior is between $\frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}$ and $\frac{1}{2}$ and whose right interim posterior above $\frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}$, would find opposite-biased learning optimal.

To see this, by observation 1,

$$EU(\sigma^L, \sigma^e|p_0) - EU(\sigma^R, \sigma^e|p_0) = \mathbb{P}(l^e|p_0)(EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e)))+$$

$$\mathbb{P}(r^e|p_0)(EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e)))$$

and if the left interim posterior is below $\frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}$ and the right interim posterior is between $\frac{1}{2}$ and $\frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}$

$$EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e)) = 0, \;\; EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e)) < 0 \implies$$

$$EU(\sigma^L, \sigma^e|p_0) - EU(\sigma^R, \sigma^e|p_0) < 0.$$

while if the left interim posterior is between $\frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}$ and $\frac{1}{2}$ and whose right interim posterior above $\frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}$

$$EU(\sigma^L|p(l^e)) - EU(\sigma^R|p(l^e)) > 0, \;\; EU(\sigma^L|p(r^e)) - EU(\sigma^R|p(r^e)) = 0 \implies$$

$$EU(\sigma^L, \sigma^e|p_0) - EU(\sigma^R, \sigma^e|p_0) > 0.$$

For the left interim posterior is below $\frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}$ and the right interim posterior is between $\frac{1}{2}$ and $\frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}$, two conditions are required:

$$p(l^e) \in \left[0, \frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}\right] \text{ and } p(r^e) \in \left[\frac{1}{2}, \frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}\right] \iff$$

$$p_0 \in \left[0, \frac{\pi^L(r|L)\pi^e(l|L)}{\pi^L(r|L)\pi^e(l|L)+\pi^L(r|R)\pi^e(l|R)}\right] \text{ and } p_0 \in \left[p_2, \frac{\pi^R(l|L)\pi^e(r|L)}{\pi^R(l|L)\pi^e(r|L)+\pi^R(l|R)\pi^e(r|R)}.\right]$$

And for these two conditions to be met at the same time it should be that

$$p_2 \leq \frac{\pi^L(r|L)\pi^e(l|L)}{\pi^L(r|L)\pi^e(l|L)+\pi^L(r|R)\pi^e(l|R)} \iff \frac{\pi^e(l|L)}{\pi^e(l|R)}\frac{\pi^e(r|R)}{\pi^e(r|L)} \geq \frac{\pi^L(r|R)}{\pi^L(r|L)}.$$

For the left interim posterior is between $\frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}$ and $\frac{1}{2}$ and whose right interim posterior above $\frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}$ other two conditions are required:

$$p(l^e) \in \left[\frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)}, \frac{1}{2}\right] \text{ and } p(r^e) \in \left[\frac{\pi^R(l|L)}{\pi^R(l|L)+\pi^R(l|R)}, 1\right] \iff$$

$$p_0 \in \left[\frac{\pi^L(r|L)\pi^e(l|L)}{\pi^L(r|L)\pi^e(l|L)+\pi^L(r|R)\pi^e(l|R)}, p_5\right] \text{ and } p_0 \in \left[\frac{\pi^R(l|L)\pi^e(r|L)}{\pi^R(l|L)\pi^e(r|L)+\pi^R(l|R)\pi^e(r|R)}, 1\right].$$

And for these two conditions to be met at the same time it should be that

$$\frac{\pi^R(l|L)\pi^e(r|L)}{\pi^R(l|L)\pi^e(r|L)+\pi^R(l|R)\pi^e(r|R)} \leq p_5 \iff \frac{\pi^e(l|L)}{\pi^e(l|R)}\frac{\pi^e(r|R)}{\pi^e(r|L)} \geq \frac{\pi^R(l|L)}{\pi^R(l|R)}.$$

Note that $\frac{\pi^R(l|L)}{\pi^R(l|R)} = \frac{\pi^L(r|R)}{\pi^L(r|L)}$ by symmetry and lemma 1. Therefore, the conditions for $p_2 \leq \frac{\pi^L(r|L)\pi^e(l|L)}{\pi^L(r|L)\pi^e(l|L)+\pi^L(r|R)\pi^e(l|R)}$ and $\frac{\pi^R(l|L)\pi^e(r|L)}{\pi^R(l|L)\pi^e(r|L)+\pi^R(l|R)\pi^e(r|R)} \leq p_4$ are equivalent.[16]

Under such conditions, we can let $p_3 = \min\left\{\frac{\pi^L(r|L)\pi^e(l|L)}{\pi^L(r|L)\pi^e(l|L)+\pi^L(r|R)\pi^e(l|R)}, \frac{\pi^R(l|L)\pi^e(r|L)}{\pi^R(l|L)\pi^e(r|L)+\pi^R(l|R)\pi^e(r|R)}\right\}$

[16]Note that this is a weaker condition than $\min\{\frac{\pi^e(l|L)}{\pi^e(l|R)}, \frac{\pi^e(r|R)}{\pi^e(r|L)}\} \geq \frac{\pi^L(r|R)}{\pi^L(r|L)}$.

and $p_4 = \max\left\{\frac{\pi^L(r|L)\pi^e(l|L)}{\pi^L(r|L)\pi^e(l|L)+\pi^L(r|R)\pi^e(l|R)}, \frac{\pi^R(l|L)\pi^e(r|L)}{\pi^R(l|L)\pi^e(r|L)+\pi^R(l|R)\pi^e(r|R)}\right\}$. Otherwise, $p_3 = p_2$ and $p_4 = p_5$. Note the effect of the informativeness of the exogenous source in these thresholds is ambiguous. ■

The expression for the expected utility from a given binary external signal, σ^e and one chosen signal, σ^x, will be useful for proving the Proposition **??**. It looks as follows:

$$Eu(\sigma^e, \sigma^x|p_0) =$$

$$p_0[\pi^e(l|R)\pi^x(l|R)u_{l^el^x|R}+\pi^e(r|R)\pi^x(l|R)u_{r^el^x|R}+\pi^e(l|R)\pi^x(r|R)u_{l^er^xR|R}+\pi^e(r|R)\pi^x(r|R)u_{r^er^x|R}]$$

$$+(1-p_0)[\pi^e(l|L)\pi^x(l|L)u_{l^el^x|L}+\pi^e(r|L)\pi^x(l|L)u_{r^el^x|L}+\pi^e(l|L)\pi^x(r|L)u_{l^er^x|L}+\pi^e(r|L)\pi^x(r|L)u_{r^er^x|L}]$$

where $u_{m^eyn^x|\theta}$ is the payoff from receiving a message m from σ^e and a message n from σ^x given that the state is θ. For instance,

$$u_{l^el^x|R} = \mathbb{1}\{p(l^e, l^e) > \frac{1}{2}\} = \mathbb{1}\{\frac{\pi^e(l|R)\pi^x(l|R)p_0}{\pi^e(l|R)\pi^x(l|R)p_0 + \pi^e(l|L)\pi^x(l|L)(1-p_0)} > \frac{1}{2}\}$$

and $u_{l^el^x|L} = \mathbb{1}\{p(l^e, l^e) < \frac{1}{2}\} = 1 - u_{l^el^x|R}$.

Using this, the following proof shows a less general version of Proposition 2, when $\sigma_e \in \{sigma^L, \sigma^R\}$. The new and complete proof is pending to be added, but follows a similar logic.

Proof.

First, let us look at the case where σ^L and σ^R are symmetric and $\sigma^e = \sigma^L$. In that case, all the possible (final) posteriors are:

$$p(l^L, l^L) = \frac{\pi^L(l|R)^2 p_0}{\pi^L(l|R)^2 p_0 + \pi^L(l|L)^2(1-p_0)},$$

$$p(l^L, r^L) = \frac{\pi^L(l|R)\pi^L(r|R)p_0}{\pi^L(l|R)\pi^L(r|R)p_0 + \pi^L(l|L)\pi^L(r|L)(1-p_0)},$$

$$p(r^L, r^L) = \frac{\pi^L(r|R)^2 p_0}{\pi^L(r|R)^2 p_0 + \pi^L(r|L)^2(1-p_0)} \text{ and}$$

$$p(l^L, l^R) = \frac{\pi^L(l|R)\pi^R(l|R)p_0}{\pi^L(l|R)\pi^R(l|R)p_0 + \pi^L(l|L)\pi^R(l|L)(1-p_0)},$$

$$p(l^L, r^R) = p(r^L, l^R) = p_0,$$

$$p(r^L, r^R) = \frac{\pi^L(r|R)\pi^R(r|R)p_0}{\pi^L(r|R)\pi^R(r|R)p_0 + \pi^L(r|L)\pi^R(r|L)(1-p_0)}.$$

Note that $p(l^L, l^R) < p(l^L, l^L) < p(l^L, r^R) < p(l^L, r^L) < p(r^L, r^R) < p(r^L, r^L)$. Using the above expression for $Eu(\sigma^e, \sigma^x|p_0)$ one can derive the following:

$$Eu(\sigma^L, \sigma^L|p_0) - Eu(\sigma^L, \sigma^R|p_0) =$$

$$\begin{cases} 0, & p(r^L, r^L) < \frac{1}{2} \\ p_0\pi^L(r|R)^2 - (1-p_0)\pi^L(r|L)^2, & p(r^L, r^R) < \frac{1}{2} < p(r^L, r^L) \\ (\pi^L(l|L) - \pi^L(r|R))((1-p_0)\pi^L(r|L) - p_0\pi^L(r|R)), & p(l^L, r^L) < \frac{1}{2} < p(r^L, r^R) \\ p_0\pi^L(r|R)(\pi^L(r|L) + \pi^L(l|R)) - (1-p_0)\pi^L(r|L)(\pi^L(l|L) + \pi^L(r|R)), & p(l^L, r^R) < \frac{1}{2} < p(l^L, r^L) \\ (\pi^L(l|L) - \pi^L(r|R))((1-p_0)\pi^L(l|L) - p_0\pi^L(l|R)), & p(l^L, l^L) < \frac{1}{2} < p(l^L, r^R) \\ p_0\pi^L(l|R)\pi^L(r|L) - (1-p_0)\pi^L(l|L)\pi^L(r|R), & p(l^L, l^R) < \frac{1}{2} < p(l^L, l^L) \\ 0, & p(l^L, l^R) > \frac{1}{2} \end{cases}$$

In addition, the different cases can be rewritten in terms of the DM's prior, that is,

$$Eu(\sigma^L, \sigma^L|p_0) - Eu(\sigma^L, \sigma^R|p_0) =$$

$$\begin{cases} 0, & p_0 < \frac{\pi^L(r|L)^2}{\pi^L(r|L)^2+\pi^L(r|R)^2} \\ p_0\pi^L(r|R)^2 - (1-p_0)\pi^L(r|L)^2, & \frac{\pi^L(r|L)^2}{\pi^L(r|L)^2+\pi^L(r|R)^2} < p_0 < \frac{\pi^L(r|L)\pi^L(l|R)}{\pi^L(r|L)\pi^L(l|R)+\pi^L(r|R)\pi^L(l|L)} \\ (\pi^L(l|L) - \pi^L(r|R))((1-p_0)\pi^L(r|L) - p_0\pi^L(r|R)), & \frac{\pi^L(r|L)\pi^L(l|R)}{\pi^L(r|L)\pi^L(l|R)+\pi^L(r|R)\pi^L(l|L)} < p_0 < \frac{\pi^L(l|L)\pi^L(r|L)}{\pi^L(l|L)\pi^L(r|L)+\pi^L(l|R)\pi^L(r|R)} \\ p_0\pi^L(r|R)(\pi^L(r|L) + \pi^L(l|R)) - (1-p_0)\pi^L(r|L)(\pi^L(l|L) + \pi^L(r|R)), & \frac{\pi^L(l|L)\pi^L(r|L)}{\pi^L(l|L)\pi^L(r|L)+\pi^L(l|R)\pi^L(r|R)} < p_0 < \frac{1}{2} \\ (\pi^L(l|L) - \pi^L(r|R))((1-p_0)\pi^L(l|L) - p_0\pi^L(l|R)), & \frac{1}{2} < p_0 < \frac{\pi^L(l|L)^2}{\pi^L(l|L)^2+\pi^L(l|R)^2} \\ p_0\pi^L(l|R)\pi^L(r|L) - (1-p_0)\pi^L(l|L)\pi^L(r|R), & \frac{\pi^L(l|L)^2}{\pi^L(l|L)^2+\pi^L(l|R)^2} < p_0 < \frac{\pi^L(l|L)\pi^L(r|R)}{\pi^L(l|L)\pi^L(r|R)+\pi^L(l|R)\pi^L(r|L)} \\ 0, & p_0 > \frac{\pi^L(l|L)\pi^L(r|R)}{\pi^L(l|L)\pi^L(r|R)+\pi^L(l|R)\pi^L(r|L)} \end{cases}$$

From this, it is clear that when $\sigma_e = \sigma^L$ a DM with prior above p_5 or below p_1 is indifferent between σ^L and σ^R.

Let us consider the rest of the cases. When $p_0 \in \left(\frac{\pi^L(r|L)^2}{\pi^L(r|L)^2+\pi^L(r|R)^2}, \frac{\pi^L(r|L)\pi^L(l|R)}{\pi^L(r|L)\pi^L(l|R)+\pi^L(r|R)\pi^L(l|L)}\right)$, the DM's optimal choice is σ^L (own-biased learning) since

$$p_0\pi^L(r|R)^2 - (1-p_0)\pi^L(r|L)^2 > 0 \iff p_0 > \frac{\pi^L(r|L)^2}{\pi^L(r|L)^2 + \pi^L(r|R)^2}.$$

Similarly, when $p_0 \in \left(\frac{\pi^L(l|L)^2}{\pi^L(l|L)^2+\pi^L(l|R)^2}, \frac{\pi^L(l|L)\pi^L(r|R)}{\pi^L(l|L)\pi^L(r|R)+\pi^L(l|R)\pi^L(r|L)}\right)$, the DM's optimal choice is σ^R since

$$p_0\pi^L(l|R)\pi^L(r|L) - (1-p_0)\pi^L(l|L)\pi^L(r|R) < 0 \iff p_0 < \frac{\pi^L(l|L)\pi^L(r|R)}{\pi^L(l|L)\pi^L(r|R) + \pi^L(l|R)\pi^L(r|L)}.$$

When $p_0 \in \left[\frac{\pi^L(r|L)\pi^L(l|R)}{\pi^L(r|L)\pi^L(l|R)+\pi^L(r|R)\pi^L(l|L)}, \frac{\pi^L(l|L)\pi^L(r|L)}{\pi^L(l|L)\pi^L(r|L)+\pi^L(l|R)\pi^L(r|R)}\right)$,

$$(\pi^L(l|L) - \pi^L(r|R))((1-p_0)\pi^L(r|L) - p_0\pi^L(r|R)) > 0 \iff p_0 < \frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)} = p_2.$$

Note that $\frac{\pi^L(r|L)}{\pi^L(r|L)+\pi^L(r|R)} \in \left(\frac{\pi^L(r|L)\pi^L(l|R)}{\pi^L(r|L)\pi^L(l|R)+\pi^L(r|R)\pi^L(l|L)}, \frac{\pi^L(l|L)\pi^L(r|L)}{\pi^L(l|L)\pi^L(r|L)+\pi^L(l|R)\pi^L(r|R)}\right)$, thus, the threshold is relevant.

And by the same logic, when $p_0 \in \left(\frac{1}{2}, \frac{\pi^L(l|L)^2}{\pi^L(l|L)^2+\pi^L(l|R)^2}\right]$,

$$(\pi^L(l|L) - \pi^L(r|R))((1-p_0)\pi^L(l|L) - p_0\pi^L(l|R)) > 0 \iff p_0 < \frac{\pi^L(l|L)}{\pi^L(l|L)+\pi^L(l|R)} = p_4$$

where $\frac{\pi^L(l|L)}{\pi^L(l|L)+\pi^L(l|R)} \in \left(\frac{1}{2}, \frac{\pi^L(l|L)^2}{\pi^L(l|L)^2+\pi^L(l|R)^2}\right)$.

Finally, when $p_0 \in \left[\frac{\pi^L(l|L)\pi^L(r|L)}{\pi^L(l|L)\pi^L(r|L)+\pi^L(l|R)\pi^L(r|R)}, \frac{1}{2}\right]$,

$$p_0\pi^L(r|R)(\pi^L(r|L)+\pi^L(l|R)) - (1-p_0)\pi^L(r|L)(\pi^L(l|L)+\pi^L(r|R)) > 0 \iff$$

$$p_0 > \frac{\pi^L(r|L)(\pi^L(r|R)+\pi^L(l|L))}{\pi^L(r|R)(\pi^L(r|L)+\pi^L(l|R))+\pi^L(r|L)(\pi^L(r|R)+\pi^L(l|L))} = p_3 \text{ and}$$

$$\frac{\pi^L(r|L)(\pi^L(r|R)+\pi^L(l|L))}{\pi^L(r|R)(\pi^L(r|L)+\pi^L(l|R))+\pi^L(r|L)(\pi^L(r|R)+\pi^L(l|L))} \in \left(\frac{\pi^L(l|L)\pi^L(r|L)}{\pi^L(l|L)\pi^L(r|L)+\pi^L(l|R)\pi^L(r|R)}, \frac{1}{2}\right).$$

Now, let us look at the case where σ^L and σ^R are symmetric and $\sigma^e = \sigma^R$. In that case, all the possible (final) posteriors are:

$$p(l^R, l^R) = \frac{\pi^R(l|R)^2 p_0}{\pi^R(l|R)^2 p_0 + \pi^R(l|L)^2(1-p_0)},$$

$$p(l^R, r^R) = \frac{\pi^R(l|R)\pi^R(r|R)p_0}{\pi^R(l|R)\pi^R(r|R)p_0 + \pi^R(l|L)\pi^R(r|L)(1-p_0)},$$

$$p(r^R, r^R) = \frac{\pi^R(r|R)^2 p_0}{\pi^R(r|R)^2 p_0 + \pi^R(r|L)^2(1-p_0)} \text{ and}$$

$$p(l^L, l^R) = \frac{\pi^L(l|R)\pi^R(l|R)p_0}{\pi^L(l|R)\pi^R(l|R)p_0 + \pi^L(l|L)\pi^R(l|L)(1-p_0)},$$

$$p(l^L, r^R) = p(r^L, l^R) = p_0,$$

$$p(r^L, r^R) = \frac{\pi^L(r|R)\pi^R(r|R)p_0}{\pi^L(r|R)\pi^R(r|R)p_0 + \pi^L(r|L)\pi^R(r|L)(1-p_0)}.$$

Note that $p(l^R, l^R) < p(l^R, l^L) < p(l^R, r^R) < p(r^R, l^L) < p(r^R, r^R) < p(r^R, r^L)$. Using the above expression for $Eu(\sigma^e, \sigma^x|p_0)$ one can derive the following:

$$Eu(\sigma^L, \sigma^R|p_0) - Eu(\sigma^R, \sigma^R|p_0) =$$

$$\begin{cases} 0, & p(r^R, r^L) < \frac{1}{2} \\ p_0\pi^L(r|R)\pi^L(l|L) - (1-p_0)\pi^L(r|L)\pi^L(l|R), & p(r^R, r^R) < \frac{1}{2} < p(r^R, r^L) \\ (\pi^L(l|L) - \pi^L(r|R))((1-p_0)\pi^L(l|R) - p_0\pi^L(l|L)), & p(r^R, l^L) < \frac{1}{2} < p(r^R, r^R) \\ p_0\pi^L(r|L)(\pi^L(r|R) + \pi^L(l|L)) - (1-p_0)\pi^L(r|R)(\pi^L(r|L) + \pi^L(l|R)), & p(l^R, r^R) < \frac{1}{2} < p(r^R, l^L) \\ (\pi^L(l|L) - \pi^L(r|R))((1-p_0)\pi^L(r|R) - p_0\pi^L(r|L)), & p(l^R, l^L) < \frac{1}{2} < p(l^R, r^R) \\ p_0\pi^L(r|L)^2 - (1-p_0)\pi^L(r|R)^2, & p(l^R, l^R) < \frac{1}{2} < p(l^R, l^L) \\ 0, & p(l^R, l^R) > \frac{1}{2} \end{cases}$$

In addition, the different cases can be rewritten in terms of the DM's prior, that is,

$$Eu(\sigma^L, \sigma^R|p_0) - Eu(\sigma^R, \sigma^R|p_0) =$$

$$\begin{cases} 0, & p_0 < \frac{\pi^L(r|L)\pi^L(l|R)}{\pi^L(r|L)\pi^L(l|R)+\pi^L(r|R)\pi^L(l|L)} \\ p_0\pi^L(r|R)\pi^L(l|L) - (1-p_0)\pi^L(r|L)\pi^L(l|R), & \frac{\pi^L(r|L)\pi^L(l|R)}{\pi^L(r|L)\pi^L(l|R)+\pi^L(r|R)\pi^L(l|L)} < p_0 < \frac{\pi^L(l|R)^2}{\pi^L(l|R)^2+\pi^L(l|L)^2} \\ (\pi^L(l|L) - \pi^L(r|R))((1-p_0)\pi^L(l|R) - p_0\pi^L(l|L)), & \frac{\pi^L(l|R)^2}{\pi^L(l|R)^2+\pi^L(l|L)^2} < p_0 < \frac{1}{2} \\ p_0\pi^L(r|L)(\pi^L(r|R) + \pi^L(l|L)) - (1-p_0)\pi^L(r|R)(\pi^L(r|L) + \pi^L(l|R)), & \frac{1}{2} < p_0 < \frac{\pi^L(l|R)\pi^L(r|R)}{\pi^L(l|R)\pi^L(r|R)+\pi^L(l|L)\pi^L(r|L)} \\ (\pi^L(l|L) - \pi^L(r|R))((1-p_0)\pi^L(r|R) - p_0\pi^L(r|L)), & \frac{\pi^L(l|R)\pi^L(r|R)}{\pi^L(l|R)\pi^L(r|R)+\pi^L(l|L)\pi^L(r|L)} < p_0 < \frac{\pi^L(l|L)\pi^L(r|R)}{\pi^L(l|L)\pi^L(r|R)+\pi^L(l|R)\pi^L(r|L)} \\ p_0\pi^L(r|L)^2 - (1-p_0)\pi^L(r|R)^2, & \frac{\pi^L(l|L)\pi^L(r|R)}{\pi^L(l|L)\pi^L(r|R)+\pi^L(l|R)\pi^L(r|L)} < p_0 < \frac{\pi^L(r|R)^2}{\pi^L(r|R)^2+\pi^L(r|L)^2} \\ 0, & p_0 > \frac{\pi^L(r|R)^2}{\pi^L(r|R)^2+\pi^L(r|L)^2} \end{cases}$$

As before, from looking at this expression, it is easy to see that when $\sigma_e = \sigma^R$ a DM with prior above p_5 or below $p_1$is indifferent between any choice of σ_i. Now, let us consider the rest of the cases.

When $p_0 \in \left(\frac{\pi^L(r|L)\pi^L(l|R)}{\pi^L(r|L)\pi^L(l|R)+\pi^L(r|R)\pi^L(l|L)}, \frac{\pi^L(l|R)^2}{\pi^L(l|R)^2+\pi^L(l|L)^2}\right)$ or $p_0 \in \left(\frac{\pi^L(l|L)\pi^L(r|R)}{\pi^L(l|L)\pi^L(r|R)+\pi^L(l|R)\pi^L(r|L)}, \frac{\pi^L(r|R)^2}{\pi^L(r|R)^2+\pi^L(r|L)^2}\right)$, the DM's optimal choice is own-biased learning since

$$p_0\pi^L(r|R)\pi^L(l|L) - (1-p_0)\pi^L(r|L)\pi^L(l|R) > 0 \iff p_0 > \frac{\pi^L(r|L)\pi^L(l|R)}{\pi^L(r|L)\pi^L(l|R) + \pi^L(r|R)\pi^L(l|L)},$$

$$p_0\pi^L(r|L)^2 - (1-p_0)\pi^L(r|R)^2 < 0 \iff p_0 < \frac{\pi^L(r|R)^2}{\pi^L(r|R)^2 + \pi^L(r|L)^2} \text{ and}$$

$$\frac{\pi^L(l|R)^2}{\pi^L(l|R)^2 + \pi^L(l|L)^2} < \frac{1}{2} < \frac{\pi^L(l|L)\pi^L(r|R)}{\pi^L(l|L)\pi^L(r|R) + \pi^L(l|R)\pi^L(r|L)}.$$

For $p_0 \in \left[\frac{\pi^L(l|R)^2}{\pi^L(l|R)^2+\pi^L(l|L)^2}, \frac{1}{2}\right)$,

$$(\pi^L(l|L) - \pi^L(r|R))((1-p_0)\pi^L(l|R) - p_0\pi^L(l|L)) > 0 \iff p_0 < \frac{\pi^L(l|R)}{\pi^L(l|R) + \pi^L(l|L)} = p_2.$$

Note that $\frac{\pi^L(l|R)}{\pi^L(l|R)+\pi^L(l|L)} \in \left(\frac{\pi^L(l|R)^2}{\pi^L(l|R)^2+\pi^L(l|L)^2}, \frac{1}{2}\right)$, thus, the threshold matters.

Similarly, for $p_0 \in \left(\frac{\pi^L(l|R)\pi^L(r|R)}{\pi^L(l|R)\pi^L(r|R)+\pi^L(l|L)\pi^L(r|L)}, \frac{\pi^L(l|L)\pi^L(r|R)}{\pi^L(l|L)\pi^L(r|R)+\pi^L(l|R)\pi^L(r|L)}\right]$,

$$(\pi^L(l|L) - \pi^L(r|R))((1-p_0)\pi^L(r|R) - p_0\pi^L(r|L)) > 0 \iff p_0 < \frac{\pi^L(r|R)}{\pi^L(r|R) + \pi^L(r|L)} = p_4$$

where $\frac{\pi^L(r|R)}{\pi^L(r|R)+\pi^L(r|L)} \in \left(\frac{\pi^L(l|R)\pi^L(r|R)}{\pi^L(l|R)\pi^L(r|R)+\pi^L(l|L)\pi^L(r|L)}, \frac{\pi^L(l|L)\pi^L(r|R)}{\pi^L(l|L)\pi^L(r|R)+\pi^L(l|R)\pi^L(r|L)}\right)$.

Lastly, when $p_0 \in \left[\frac{1}{2}, \frac{\pi^L(l|R)\pi^L(r|R)}{\pi^L(l|R)\pi^L(r|R)+\pi^L(l|L)\pi^L(r|L)}\right]$,

$$p_0\pi^L(r|L)(\pi^L(r|R) + \pi^L(l|L)) - (1-p_0)\pi^L(r|R)(\pi^L(r|L) + \pi^L(l|R)) > 0 \iff$$

$$p_0 > \frac{\pi^L(r|R)(\pi^L(r|L) + \pi^L(l|R))}{\pi^L(r|R)(\pi^L(r|L) + \pi^L(l|R)) + \pi^L(r|L)(\pi^L(r|R) + \pi^L(l|L))} = p_3 \text{ and}$$

$$\frac{\pi^L(r|R)(\pi^L(r|L) + \pi^L(l|R))}{\pi^L(r|R)(\pi^L(r|L) + \pi^L(l|R)) + \pi^L(r|L)(\pi^L(r|R) + \pi^L(l|L))} \in \left(\frac{1}{2}, \frac{\pi^L(l|R)\pi^L(r|R)}{\pi^L(l|R)\pi^L(r|R) + \pi^L(l|L)\pi^L(r|L)}\right).$$

■

The proof for Proposition 7 and the proofs of propositions in Section 6 are pending to be added.